

# Lecture Topics

## I Topological Symmetries in QFT: Quiche representation

- 1 Extracting the symmetries out of the QFT
- 2 Higher symmetries and topological defects
- 3 Defect Calculus via links
- 4 Low-dimensional examples:  
algebras, fusion categories, BFCs
- 5 Framings and orientations

## II Defects in Finite Homotopy theories

- 1 Homotopy Theory vs TQFT
- 2 Defects internal to the TQFT
- 3  $(X, *)$  as a quiche
- 4 Dirichlet boundary and Dirichlet ends of defects
- 5 Condensation.
- 6 Examples: Condensed defects in the fht case
- 7 Conjectural classification: fhts, EM duality,  
anomaly cancellation

## III Condensation and Unlinking

- 1 Heuristic Description
- 2 Main Conjecture and corrected versions
- 3 Proof of one version of the conjecture

# I Topological Symmetries in QFT

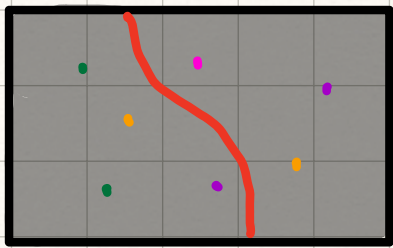
1 Our QFT will be Wick-rotated to Euclidean time.

A useful and comprehensive definition of QFT is not available except in low dimensions ( $n=1$ , Quantum Mechanics, &  $n=2$ , especially CFT. See work on Conformal Nets.)

But in any case a QFT  $\mathcal{F}$  is "something" that produces numbers ("correlation functions") for Riemannian  $n$ -manifolds with "operator insertions" at points.

Remark We must also allow for extended operators with positive-dimensional support, as we'll see for topological operators. [Eg. Wilson Lines].

A symmetry of  $\mathcal{F}$  is an interface between isomorphic copies of  $\mathcal{F}$  on the two sides but with a non-trivial identification.



Eg a transformation of the fields that preserves the Lagrangian

This is often called topological if its placement can be freely moved (while avoiding the insertion fields) without changing the correlators.

The quiche (or SymTFT) is a device for implementing topological symmetry in a QFT while taking the symmetry calculus outside.

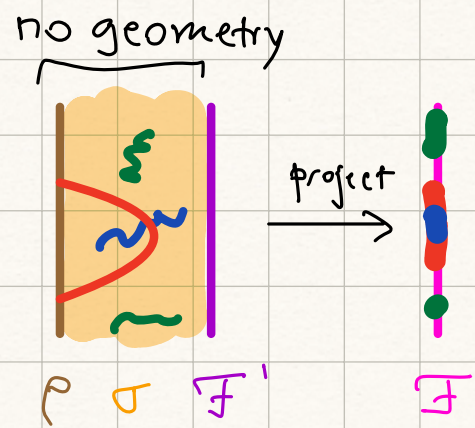
## Basic picture

$\sigma$  an  $(n+1)$ -dim TQFT

$\rho$  a (left) topological theory

$\mathcal{F}'$  a (right) theory

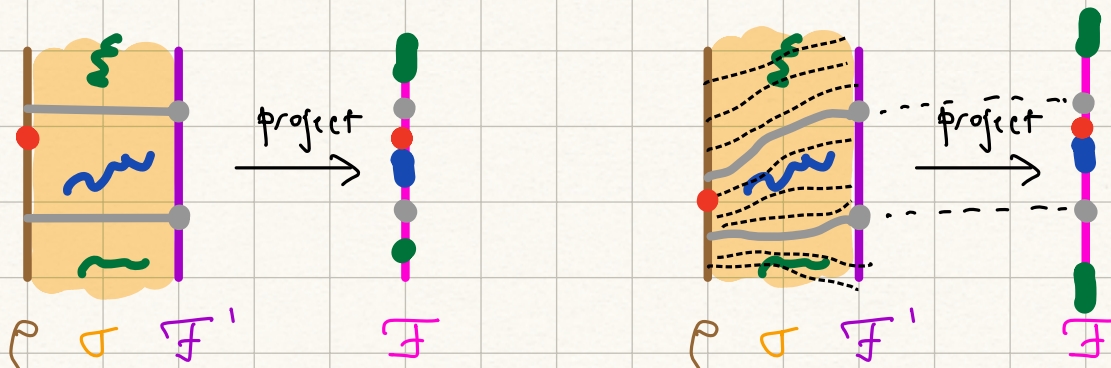
$\mathcal{F}$  the dimensional reduction



   Embedded topological operators in  $(\sigma, \rho)$  projecting to topological defects in  $\mathcal{F}$ .

Remarks (i)  $\sigma$  need not be fully defined: we don't use its top-dimensional invariant.

(ii) Operators in the bunch that touch the  $\mathcal{F}'$  boundary are usually **NOT** topological. The topological nature of the others is manifested by isotoping the projection



The projection may be moved provided we do not move the extended non-topological defects — over the others: this would change the internal calculus

But green, blue defects can move freely: they commute  
Red defects are confined to the left boundary.  
In this case, they may not commute.

# Paradigm (QM)

$\mathcal{F}$  = vector space (with Hamiltonian  $H$ )

$\sigma$  = algebra  $A \subset \text{End}(\mathcal{F})$  commuting with  $H$

$\mathcal{F}' = \mathcal{F}$  as  $A$ -module

$\rho = A$  as right module

Projection:  $\mathcal{F} \cong \mathcal{F}' \otimes_A A$

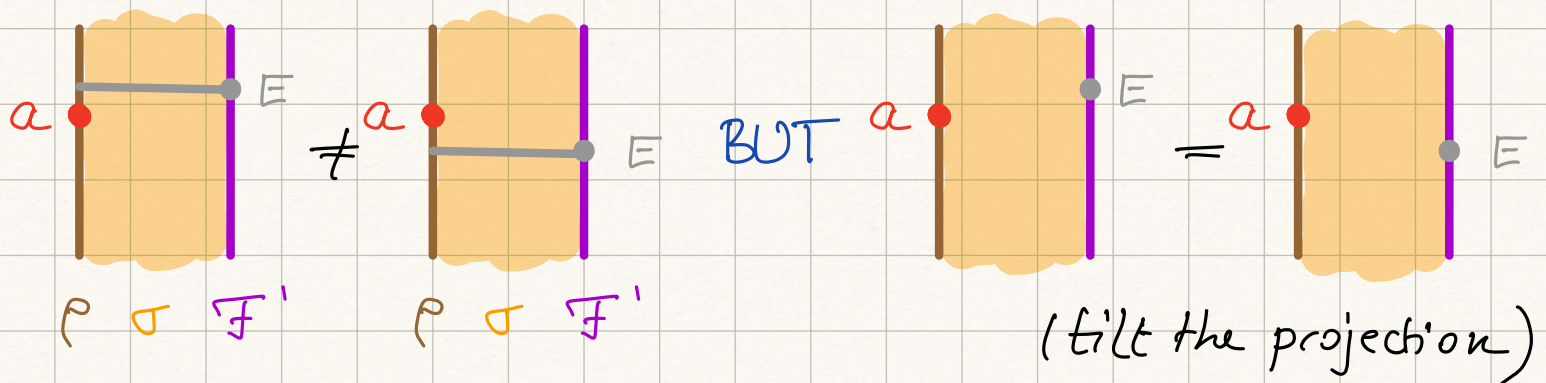
green, blue defects: central elt of  $A$

Red defects: general elt of  $A$

Gray defect:  $\in \text{End}(\mathcal{F})$  Not necessarily commuting w/  $H$

Gray defect entirely on  $\mathcal{F}'$  boundary:

$\in \text{End}(\mathcal{F})$  commuting with  $A$  (not  $H$ )



Observation:  $A = \text{End}_A(A) = \boxed{\text{End}_\sigma(\rho)}$

This is the algebra of symmetries that acts on  $\mathcal{F}'$ !

Had we chosen another module  $M$  for  $A$ , the symmetries acting would be  $a \in \text{End}_A(M)$  on  $\mathcal{F}'$  and  $\mathcal{F} \cong M \otimes_A \mathcal{F}'$

Key Fact There is a precise dictionary between the topological depiction of symmetries and the algebraic representation on  $\mathcal{F}$ . Not a metaphor!

2. An algebra of topological symmetries of a QFT  $\mathcal{F}$  is a presentation as dimensional reduction of a quiche  $(\mathcal{P} | \sigma |$  and a boundary QFT  $\mathcal{F}'$ .

Remark Often,  $\sigma$  is an algebra object in a (higher) category and  $\mathcal{P}$  the regular module. Then we can say that  $\sigma$  "is" an algebra of topological symmetries of  $\mathcal{F}$ . But in general, the algebra that acts on  $\mathcal{F}$  is  $\text{End}_{\sigma}(\mathcal{P})$ .

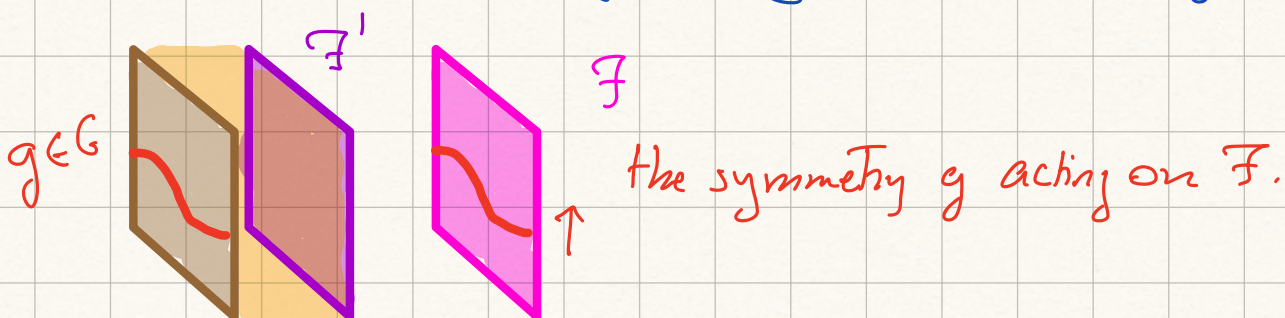
Remarks (i) Topological operators embedded in  $\sigma$  may always be pushed to the boundary  $\mathcal{P}$ . So it suffices to study those.

(ii) When  $\dim \mathcal{F} = 1$ , the algebra/module picture captures everything:  $\sigma$  could be a linear category and  $\mathcal{P}$  an object inside, what acts on  $\mathcal{F}$  is the algebra  $\text{End}_{\sigma}(\mathcal{P})$ .

### Higher (co-) dimension

When  $\dim \mathcal{F} = \dim \mathcal{P} = n$ ,  $\dim \sigma = n+1$ , extended operators may be placed in  $\mathcal{P}$ . In fact it's the codimension 1 ops that are the familiar symmetries.

Eg  $\sigma =$  finite group gauge thry,  $\mathcal{P} =$  gauge-fixing (Dirichlet) bdy



## How to think of higher codimension symmetries?

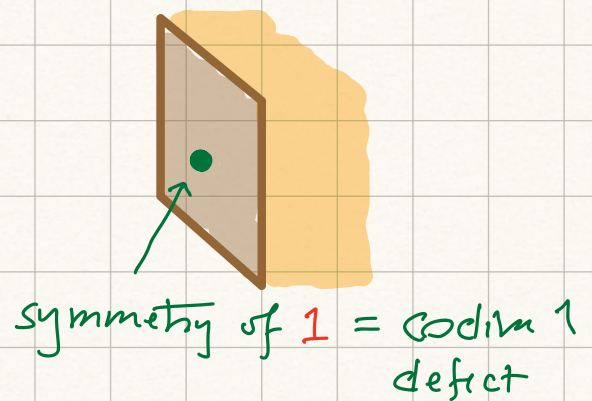
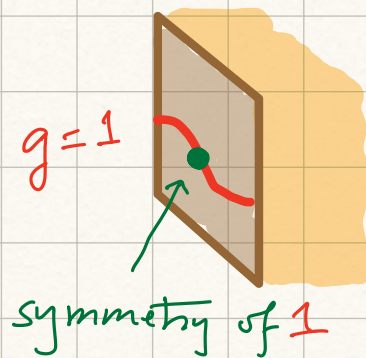
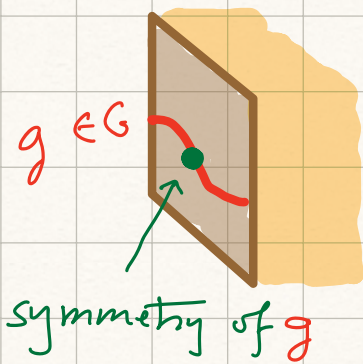
When acting on objects in a higher category, symmetries can have symmetries of their own. Example:

2-cat of (algebras, bimodules, intertwiners)

$A$ - $A$  bimodule  $M$  = "noninvertible symmetry"

endo/auto morphism of  $M$  = non/invertible symmetry of the symmetry

A special symmetry is the identity (transparent codim 1 defect)



"Noninvertible symmetries = defects = extended operators"

# Meta Conjecture from physics

Given an  $n$ -dimensional QFT  $\mathcal{F}$

its topological defects of all codimensions = tensor  $n$ -category

$\Rightarrow$  defines a guiche  $(\rho, \sigma)_{\mathcal{F}}$  (reg. moduln, algebra)

comprising all topological symmetries of  $\mathcal{F}$ .

Little chance of an approachable formulation;

we understand only fully local TQFTs, but not fully local QFTs:

$\Rightarrow$  cannot define fully local top symmetries of  $\mathcal{F}$

But it is true if  $\mathcal{F}$  is a fully local TQFT!

$\mathcal{F}: \text{Bord}^{\leq n} \longrightarrow (\text{some linear } n\text{-category } C_n)$

$\mathcal{F}(\text{pt}) \in C_n$ : fully dualizable

Cobordism Hypothesis:  $\mathcal{F}(\text{pt})$  determines  $\mathcal{F}$  on framed manifolds

If  $\mathcal{F}$  is a vectn space,  $\sigma = \text{End}(\mathcal{F})$ ,  $\rho =$  regular module

Generally  $\sigma = \mathcal{F}(\text{pt}) \boxtimes \mathcal{F}(\text{pt})^{\vee} =$  algebra object in the  
 $\rho =$  right reg module  $n\text{-cat. containing } \mathcal{F}(\text{pt})$

If  $C_n$  is "simple" (eg:  $\text{End}^{\text{tr}}_{\mathbb{I}\mathbb{d}} = \mathbb{C}$ ) then

$$(\rho | \sigma | \mathcal{F}) \equiv (\mathcal{F} | \mathbb{1} | \mathbb{1}) = \mathcal{F}$$

(so the universal guiche gives a tantology)

$$V = \langle \text{End}(V) | \text{End}(V) | V \rangle = \langle V | \mathbb{C} | \mathbb{C} \rangle$$

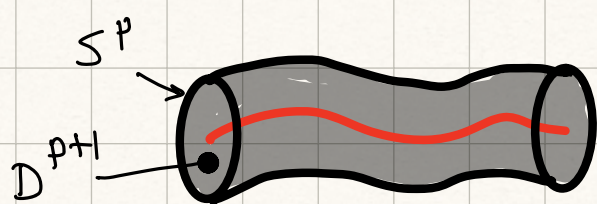
### 3 Calculus for Fully Local Defects

We work initially inside a given TQFT  $\sigma$ , before moving to defects supported on a boundary theory  $\mathcal{P}$ .

Motivation: pictures, algebraic translation cleaner in FHTs.  
(General case: Higher Hochschild homologies; Francis, Lurie)

Starting fairy tale: extended operators = embedded TQFTs

Eg:  $n$ -dim. TQFT = codim 1 defect in trivial  $(n+1)$  dim TQFT

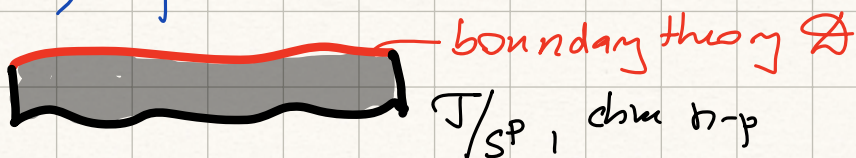


Ambient TQFT  $\mathcal{T}$ ,  $\dim. n$

defect  $\mathcal{D}$  of codim  $p+1$   
= " $p$ -fect"

Simplification (for now): framed normal bundles

Reducing along  $S^p$ :



$\mathcal{T}/S^p$ ,  $\dim. n-p$

Working definition: a  $p$ -fect is a boundary theory for  $\mathcal{T}/S^p$ .

Cobordism Hypothesis: (Lurie's Classification Theorem)

$\mathcal{T}$  defined from  $\mathcal{T}(pt) \in C_n$

$\mathcal{T}/S^p$  defined from  $\mathcal{T}(S^p) \in \Omega^p C_n := \overbrace{\text{End}(\text{End}(\dots \text{End}(\mathbb{1}) \dots))}^{p \text{ times}}_{C_n}$

$\mathcal{D}$ : interface between (trivial) and  $\mathcal{T}/S^p$

defined from  $\mathcal{D}(pt): \mathbb{1}_{\Omega^p C_n} \rightarrow \mathcal{T}(S^p)$

$\mathcal{D}(pt)$  is an object in  $\mathcal{T}(\text{link of defect})$

Example Finite gauge theory  $G$  in 3d. (2D: exercise)

[Algebraically: generated by tensor category  $\text{Vec}\langle G \rangle$ ]

Codim 1 defect:  $\text{Link} = S^0$ ,  $g(S^0) = g \boxtimes g^r$

Boundary theory = category with  $G \times G$  action ( $g^r \cong g^{\text{hom}}$ )

Simplex  $\leftrightarrow$  Subgroups  $H \subset G \times G + \mathbb{C}^*$  Central extensions

category:  $\text{Vec}^T[G \times G / H]$

Will see: they correspond to

algebra objects in anyons ( $\text{Vec}^G\langle G \rangle$ ) mod. Morita equivalence  
[Feature of 3D theories]

Codim 2 defect  $\text{Link} = S^1$ ,  $g(S^1) = \text{Vec}^G\langle G \rangle$

Two special cases:

Conjugation - equiv. vect. bndls /  $G$

Wilson defects, irreps of  $G$  supported at  $1 \in G$

't Hooft defects, line bundles on single conjugacy classes

A general line defect is a mix of these.

Codim 3 (point) defects  $g(S^2) = \mathbb{C}$ , just scalars.

Recall the partition function  $g(M^3)$  is the weighted count of bundles

Surface defects modify the count by forcing a reduction to  $H$   
on the hypersurface defect

(central extension gets integrated on surface  $\rightarrow$  weight)

Wilson defects include a factor of  $\text{Tr}(\text{holonomy})$  in weight

't Hooft defects count bundles with singularity along the loop,  
linking monodromy in conjugacy class

## Example Finite gauge theory in 4dms

Codim 1 Link  $S^0$ , 3-dim TFTs with  $G \times G$  action  
(eg: fusion cats, but also Reshetikhin-Turaev theories)

No classification;

$$G \times G \rightarrow \text{BrPic}(F) = \left\{ \begin{array}{c} \mathbb{C}^\times \\ [Z(F)^\times] \\ \text{BrAut}(Z(F)) \end{array} \right\} \begin{array}{c} \pi_2 \\ \pi_1 \\ \pi_0 \end{array}$$

Codim 2 Link  $S^1$ ,  $g(S^1) = \bigoplus_{g \in G/\text{Ad}G} (\text{Vec}\langle Z(g) \rangle, *)$

3D gauge theory for centralizer

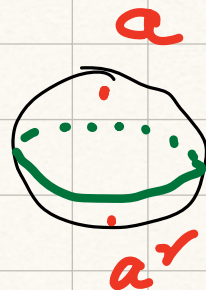
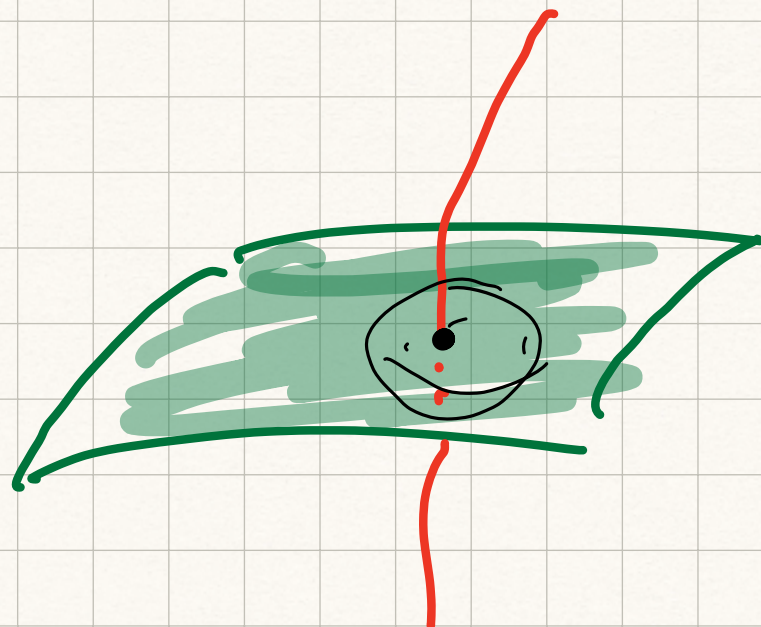
't Hooft defects use the categories  $\text{Vec}$  over a single conj class

Codim 3 Link  $S^2$ ,  $g(S^2) = \text{Rep}(G)$ , linear category  
Wilson lines

Exercise Write the partition function.

Not so easy for general surface defects!

Must blow up the defect line and treat as manifold w/ bdy.



This is a bit too simple

Just as TFTs may contain defects,  
defects may contain subdefects  
So we must iterate!

In fact the support of a defect need not be smooth;  
we can talk about **constructible defects** supported on  
stratified subspaces, locally constant on strata

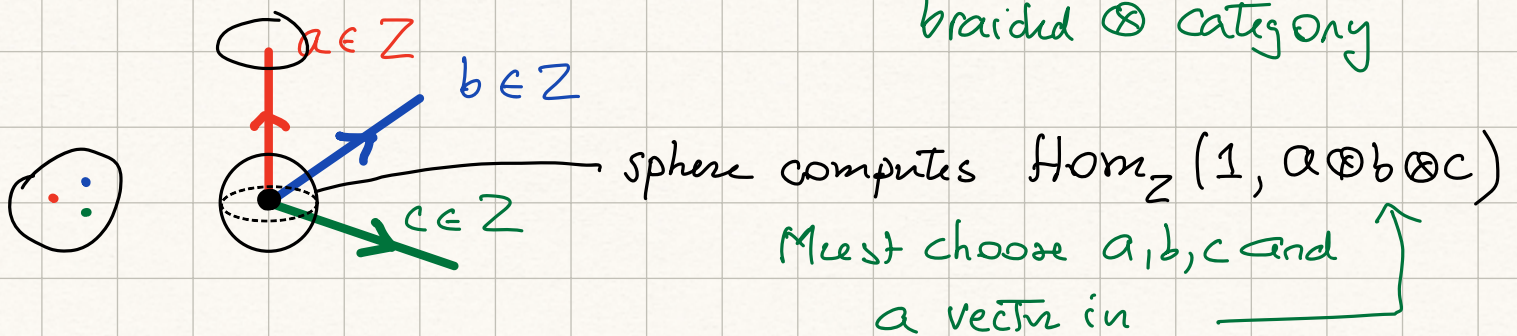
Recipe for deciding what defect may go on a given support

Find the linking spheres, from top dimension down

Determine the choices of defects & keep going.

After the top dimension, the linking spheres will contain defects!

Example 3D gauge theory;  $Z := \mathcal{G}(S') = \text{Vec}^G(G)$   
braided  $\otimes$  category



This calculus is a (non-probabilistic) version  
of the **Scattering (S-matrix) QFT Computation**

Always get probabilities 0 or 1 (possible or impossible)

## Brief side discussion: tangential structures

The Cobordism hypothesis describes fully local TQFTs

$$J: \text{Bord}_n^{\text{fr}} \longrightarrow \mathcal{C}_n \quad (\text{some } n\text{-category}) \\ \text{usually } \mathbb{C}\text{-linear})$$

But the gauge theories (and their FHT generalizations) extend to unoriented manifolds.

Tweaks can require orientations or higher (Spin) structures:

↳  $\tau \in H^n(BG; \mathbb{C}^\times)$  Dijkgraaf - Witten twist (higher extension)

Bund  $M^n \xrightarrow{f} BG$ , use  $\int_M f^* \tau \in \mathbb{C}^\times$  as a weight

$\tau \in H^n(BG; \text{super-}\mathbb{C}^\times)$ , for theories with super-vector spaces  
+  $H^{n+1}(BG; \{\pm 1\})$  but w/ restriction

But when studying defects and boundaries, restrictions on tangential structures can crop up without that.

Example (fusion categories  $F$ , eg  $\text{Vec}\langle G \rangle$  or  $\text{Vec}^\tau\langle G \rangle$ )

Theorem (Müger, Turaev, ..., Freed - Scheimbauer - T)

Promoting the Turaev - Viro 3D theory built from  $F$  to oriented manifolds requires a ribbon structure on  $Z(F)$   
+ vanishing of a sign anomaly ( $\in \text{mod } 24$ ; 0 or 12).

Ribbon: homogeneous quadratic refinement of braiding

$$\theta: \text{Ob } Z(F) \longrightarrow \mathbb{C}^\times$$

$$\beta_{x,y}: x \otimes y \xrightarrow{\sim} y \otimes x$$

$$\beta_{x,y} \circ \beta_{y,x} = \theta(x \otimes y) \theta(x)^{-1} \theta(y)^{-1}$$

$$\theta(x^\vee) = \theta(x)^\vee$$

bilinear

quadratic

homogeneity

Can change a ribbon by a multiplicative map  $Z(F) \rightarrow \{\pm 1\}$

Eg (Four orientation structures on  $\mathbb{Z}_2 \cong \{\pm 1\}$  gauge theory)

$$F = \text{Vec} \langle \pm 1 \rangle \quad Z(F) = \text{Vec}^{\langle \pm 1 \rangle} \langle \pm 1 \rangle = \langle 1, e, m, em \rangle$$

$e = \text{sign rep. (Wilson)}, m = (-1) \in \{\pm 1\}$

### 1. Standard

Count double covers

No orientation required!

2 boundary conds, Dirichlet / Neumann

4 line operators.

$$\subseteq H^2(M; \mathbb{Z}_2)$$

### 2. Gravitational

Count double covers weighted by  $\int_M \overset{\text{tangent}}{\omega_2(TM)} \cdot \overset{\text{cover}}{\omega_1(m)} = \pm 1$

{ Neumann condition needs a Spin structure on  $\partial M$ !  
(must kill  $\omega_2$  on  $\partial M$  to define the integral)  
Dirichlet condition does not

't Hooft loop  $m$  (and  $em$ ) need a Spin structure on  $TM$   
Wilson loop  $e$  does not

### 3. Spin-Counting

Count Spin structures on  $M$ .

{ Dirichlet condition IS a spin structure on  $\partial M$   
(Neumann cond requires no spin structure on  $\partial M$ )  
 $e$  and  $em$  need spin structures along loop

### 4. Spin-counting relative to a $p_1$ -structure

$$c = 12 \bmod 24$$

Given  $\tau$  with  $\delta\tau = p_1$ , Spin structure  $\omega(M)$  with  $\delta\omega(M) = \omega_2(TM)$

$\{\tau - \omega_2(TM) \cup \omega(M)\} \in H^3(M; \mathbb{Z}_2)$  a counting weight for Spin structures

Both boundary conditions require Spin structures now.