

Lecture 3

Lost two lectures: Unitary Symmetry operators, their defects, and anomalies

This lecture: generalized Symmetry on the lattice

→ Non-invertible and higher-form Symmetry

→ Currently lacks a general understanding

⇒ focus on examples.

Non-invertible Symmetry in 1+1d

Consider lattice on S^1 with L sites and for a finite group G (called G -gubits)

$$\mathcal{H}_j = \text{Span}_{\mathbb{C}} \{ |g\rangle \mid g \in G \} \quad \text{with} \quad \langle g|h \rangle = \delta_{g,h}.$$

→ $\text{Hom}(G, \text{U}(1))$ onsite Symmetry operators

$$W_{\chi} = \sum_{\vec{g}} \prod_{j=1}^L \chi(g_j) \, |\vec{g}\rangle \langle \vec{g}|$$

with $\vec{g} = (g_1, g_2, \dots, g_L)$ and $\chi \in \text{Hom}(G, \text{U}(1))$

e.g., $G = \mathbb{Z}_2$: Each $\mathcal{H}_j$ is a qubit and

$$W_{\text{triv}} = \mathbb{1} \quad W_{\text{sign}} = \prod_{j=1}^L Z_j.$$

Each χ is a 1d rep of G . What about general Unitary irrep

$$\Gamma: G \longrightarrow \text{GL}(d_{\Gamma}, \mathbb{C})$$

Let

$$W_{\Gamma} = \sum_{\vec{g}} \text{Tr}_{\Gamma} \left(\prod_{j=1}^L g_j \right) |\vec{g}\rangle \langle \vec{g}|$$

→ generalizes the W_X operators.

$$\text{Suppose } \Gamma_a \otimes \Gamma_b = \bigoplus_c N_{ab}^c \Gamma_c \quad (N_{ab}^c \in \mathbb{Z}_{\geq 0})$$

→ Follows from representation theory

$$W_{\Gamma_a} W_{\Gamma_b} = W_{\Gamma_a \otimes \Gamma_b} = \sum_c N_{ab}^c W_{\Gamma_c}.$$

⇒ $\{W_{\Gamma}\}_{\Gamma \in \text{irrep}(G)}$ are $\text{Rep}(G)$ -Symmetry operators.

Non-invertible Symmetry if G is non-abelian:

⇒ For non-abelian G , $\exists$ an irrep Γ with $d_{\Gamma} > 1$

⇒ Math fact: Every Γ with $d_{\Gamma} > 1$ has at least one $g \in G$ with $\text{Tr}_{\Gamma}(g) = 0$

⇒ W_{Γ} with Γ and $d_{\Gamma} > 1$ is non-invertible.

↳ Simple statement about a Matrix!

Example: $G = S_3 = \langle r, s \mid r^3 = s^2 = 1, srs^{-1} = r^{-1} \rangle$.

→ three irreps:

1) trivial:

$$\mathbb{1}(r) = 1$$

$$\mathbb{1}(s) = 1$$

2) sign:

$$P(r) = 1$$

$$P(s) = -1$$

3) 2d:

$$E(r) = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix} \quad E(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

→ Symmetry operators $W_L = \mathbb{1}$, W_P , W_E satisfy

$$W_P^2 = \mathbb{1} \quad W_P W_E = W_E W_P = W_E \quad W_E^2 = \mathbb{1} + W_P + W_E$$

Non-invertibility: $\text{Tr}_E(S) = 0 \Rightarrow W_E$ is non-invertible

There are infinitely many $\text{Rep}(G)$ -symmetric local Hamiltonians.

Example:

let $\vec{X}^{(h)} |g\rangle = |hg\rangle$, $\overleftarrow{X}^{(h)} |g\rangle = |gh^{-1}\rangle$, $F|g\rangle = \delta_{g,1} |g\rangle$.

$\text{Rep}(G)$ Potts model

$$H = - \sum_{j=1}^L \sum_{h \in G} \overleftarrow{X}_j^{(h)} \vec{X}_{j+1}^{(h)} - h \sum_{j=1}^L F_j$$

→ For $G = \mathbb{Z}_q$, this is the q -state Potts model
(For $q=2$, it's the Ising model)

→ $[H, W_P] = 0$ for all P .

→ $h \ll 1$: $\text{Rep}(G)$ SSB phase

$h \gg 1$: $\text{Rep}(G)$ symmetric phase.

$\text{Rep}(G)$ symmetry defects

→ dual inrop $\bar{\Gamma}(g) = \Gamma(g)^*$

We insert a pair of $\bar{\Gamma}$ Γ defects using the truncation

$$W_P^I = \sum_{\alpha, \beta, \vec{g}} \Gamma \left(\prod_{j=\partial_L}^{\partial_R} g_j \right)_{\alpha\beta} |\alpha\rangle_L \otimes |\beta\rangle_R \otimes |\vec{g}\rangle \langle \vec{g}|$$

with $I = [\partial_L, \partial_R]$ (Natural when W_P written as an MPO)

→ Not an operator of $\mathcal{H}$ for $d_r > 1$. Instead a linear map

$$\mathcal{H} \mapsto \mathcal{H}_{\text{defect}} \cong \mathcal{H} \otimes \mathbb{C}^{d_r} \otimes \mathbb{C}^{d_r}$$

Let $(W_r^{\pm})^{\dagger} = \sum_{\alpha, \beta, \vec{g}} \Gamma^* \left(\prod_{j=j_L}^{j_R} g_j \right)_{\alpha\beta} |\vec{g}\rangle \langle \alpha|_L \otimes \langle \beta|_R \otimes \langle \vec{g}|$. The

defect insertion map satisfies

$$(W_r^{\pm})^{\dagger} W_r^{\pm} = d_r \mathbb{1}_{\mathcal{H}}.$$

→ Note: $d_r^{-1/2} W_r^{\pm}$ is an isometry

→ Spacetime picture

$$(W_r^{\pm})^{\dagger} W_r^{\pm} = \begin{array}{c} \begin{array}{c} \leftarrow \\ \boxed{\quad} \\ \rightarrow \end{array} \\ \begin{array}{c} \downarrow \\ \boxed{\quad} \\ \uparrow \end{array} \end{array} \begin{array}{c} (W_r^{\pm})^{\dagger} \\ \bullet \\ W_r^{\pm} \end{array} = d_r \mathbb{1}_{\mathcal{H}}.$$

→ Lattice version of how topological defect lines in 1+1d Euclidean QFT satisfy

$$\langle \bigcirc_a \rangle = \text{quantum dim. of } a.$$

Movement operator $M_r^{\vec{j}}$ satisfies

$$(\mathbb{1}_L \otimes M_r^{\vec{j}_R}) W_r^{\pm} = W_r^{[\vec{j}_L, \vec{j}_R+1]}$$

$$\Rightarrow M_r^{\vec{j}} = \sum_{\beta, \gamma, \vec{g}} \Gamma(g_{\vec{j}_R+1})_{\gamma\beta} |\beta\rangle_R \otimes |\vec{g}\rangle \langle \gamma|_R \otimes \langle \vec{g}|$$

→ Is a unitary operator and satisfies

$$T_{\Gamma, \text{defect}} (M_n^L M_n^{L-1} \dots M_n^1) = W_{\Gamma} \quad (\text{defect} \rightarrow \text{operator})$$

Example in $\text{Rep}(G)$ Potts model

For non-invertible symmetry defects, defect Hamiltonian found by

$$W_{\Gamma}^{\pm} H = H_n^{(\pm)} W_{\Gamma}^{\pm}$$

$$\rightarrow H_n^{(\pm)} = \mathbb{1} \otimes \mathbb{1} \otimes H + \delta H_L + \delta H_R \quad \text{with, e.g.,}$$

$$\delta H_R = - \sum_{h \in G} \mathbb{1} \otimes (\bar{\Gamma}(h) - \mathbb{1}) \otimes \overleftarrow{X}_{j_R}^{(h)} \overrightarrow{X}_{j_R+1}^{(h)}$$

$\Rightarrow$ acts nontrivially on the Γ defects Hilbert space

$\text{Rep}(G)$ -Symmetry action

Let O be a local operator with support $R \subset \Lambda$.

For $R \subset I$, The action of W_{Γ} on O is

$$O \mapsto (W_{\bar{\Gamma}}^{\pm})^{\dagger} O W_{\bar{\Gamma}}^{\pm} \equiv \alpha_{\Gamma}(O)$$

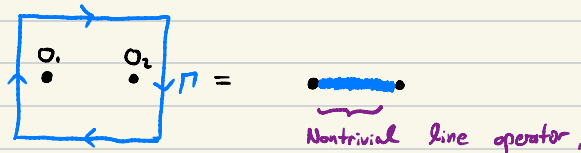
→ spacetime picture

$$(W_{\bar{\Gamma}}^{\pm})^{\dagger} O W_{\bar{\Gamma}}^{\pm} = \begin{array}{c} \begin{array}{c} \xrightarrow{\hspace{1cm}} \\ \square \\ \xleftarrow{\hspace{1cm}} \end{array} \begin{array}{c} (W_{\bar{\Gamma}}^{\pm})^{\dagger} \\ \\ W_{\bar{\Gamma}}^{\pm} \end{array} \\ \begin{array}{c} \uparrow \hspace{0.5cm} \bullet O \hspace{0.5cm} \downarrow \end{array} \end{array}$$

→ e.g., $\alpha_r(\vec{X}_j^{(g)}) = \text{Tr}_r(g) \vec{X}_j^{(g)}$ } can be mapped to zero!

→ Generally, $\alpha_r(O_1 O_2) \neq \alpha_r(O_1) \alpha_r(O_2)$

⇒ spacetime picture



Remarks

1) This $\text{Rep}(G)$ example generalizes to $\text{Rep}(H)$ Symmetry with H a finite-dim semisimple Hopf algebra
(Jia '24 ; Lu, Chatterjee, Tantivasadakarn '26)

→ Anomaly-free fusion category Symmetry.

2) Defining non-invertible symmetry operators is an active topic.

3) For systems with finite $\dim \mathcal{H}$;

→ tensor product $\mathcal{H}$: only integral fusion category symmetries can be realized (Evans, Jones '25)

quantum dims $\in \sqrt{\mathbb{Z}}$ ← Weakly integral fusion cats can be realized by mixing with G&A sym

(Inamura '26 ; Wen, Inamura, Schäfer-Nameki '26)

→ general $\mathcal{H}$: Any Unitary Fusion cat sym realizable.

(Anyon chains)

4) Every unitary Fusion cat sym realizable for tensor product $\mathcal{H}$ if $\dim \mathcal{H}$ infinite (Brunner, Jones '26)

Higher-Form Symmetry

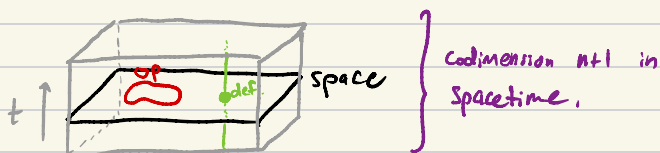
An n -form symmetry in a $(d+1)D$ Hamiltonian lattice system has

1) topological operators with support on a $d-n$ cycle:

$$\rightarrow U(C) = U(\tilde{C}) \text{ if } C \text{ homologous to } \tilde{C}$$

2) topological defects with support on a $d-n-1$ cycle. In space!

e.g., $d=2$ and $n=1$:



Higher-form symmetry operators do not exist on tensor product Hilbert spaces

Suppose $\mathcal{H} = \bigotimes_{j \in I} \mathcal{H}_j$ and C and $\tilde{C}$ have disjoint support.

$$\rightarrow U(C) = V_C \otimes \mathbb{1}_{\tilde{C}} \text{ and } U(\tilde{C}) = \mathbb{1}_C \otimes V_{\tilde{C}}$$

$$\rightarrow U(C) = U(\tilde{C}) \Rightarrow V_C \otimes \mathbb{1}_{\tilde{C}} = \mathbb{1}_C \otimes V_{\tilde{C}}$$

$$\Rightarrow V_C \propto \mathbb{1}_C \text{ and } V_{\tilde{C}} \propto \mathbb{1}_{\tilde{C}}$$

What is possible for $\bigotimes_j \mathcal{H}_j$:

$$U(C) = B_{C, \tilde{C}} U(\tilde{C})$$

with $B_{C, \tilde{C}}$ supported in region between C and $\tilde{C}$,

→ Suppose $\exists$ Mutually commuting local projectors $\{P_j\}_{j \in \Lambda}$
 s.t. $P = \prod_{j \in \Lambda} P_j$ satisfies

$$P B_{c, \tilde{c}} = B_{c, \tilde{c}} P = P \quad [P, U(c)] = 0,$$

$$\dim(\text{Im}(P)) > 0$$

$\Rightarrow$ On subspace $\text{Im}(P) \subset \mathcal{H}$: $U(c) = U(\tilde{c})$

→ $\{U(c)\}$ are higher-form symmetry operators on $\mathcal{H}_{\text{top}} = \text{Im}(P)$

$\Rightarrow \{U(c)\}$ are "lattice higher-form" symmetry operators with respect to $\{P_j\}_{j \in \Lambda}$.

(Also called non-topological, faithful, or non-relativistic higher-form symmetry operators)

→ Lattice higher-form symmetry operator can still give rise to topological defects on a tensor-product Hilbert space.

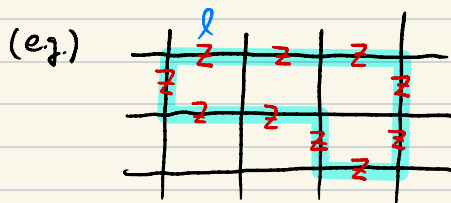
Example

Let Λ be the square lattice with a qubit on each edge

$$\mathcal{H} = \bigotimes_{e \in \Lambda} \mathbb{C}^2$$

For each lattice loop ℓ , let

$$W(\ell) = \prod_{e \in \ell} Z_e$$



→ Satisfies $\mathbb{Z}_2$ group law $W(\ell)^2 = 1$.

Denote by

Picture means take product!

$$B_p = \begin{array}{|c|} \hline \mathbb{Z} \\ \hline p \\ \hline \mathbb{Z} \\ \hline \end{array}$$

Projector to $B_p = 1$ states.

$$P_p = \frac{1}{2} (1 + B_p)$$

→ Every $W(\ell)$ satisfies

$$W(\ell + \partial \Gamma) = \left(\prod_{p \in \Gamma} B_p \right) W(\ell)$$

⇒ $\{W(\ell)\}$ are $\mathbb{Z}_2$ 1-form symmetry operators on

$$\mathcal{H}_{\text{top}} = \text{Im } \prod_p P_p = \{ |\psi\rangle \in \mathcal{H} \mid B_p |\psi\rangle = |\psi\rangle \}$$

⇒ $\{W(\ell)\}$ are $\mathbb{Z}_2$ lattice 1-form symmetry operators with respect to $\{B_p\}$.

The most general $\mathbb{Z}_2$ lattice 1-form symmetry operator here takes the form

$$W(\gamma) = \prod_{e \in \gamma} \mathbb{Z}_e^{\gamma_e}$$

→ $\mathbb{Z}_2$ -valued 1-cycle.

$$\gamma = \sum_e \gamma_e [e] \in \mathbb{Z}_1(\gamma; \mathbb{Z}_2)$$

→ The projector $\prod_p P_p$ implements quotient

$$\mathbb{Z}_1(\gamma; \mathbb{Z}_2) \longrightarrow H_1(\gamma; \mathbb{Z}_2)$$

Symmetric Hamiltonian: Toric code

$$H = - \sum_s A_s - \sum_p B_p \quad \text{with} \quad A_s = \begin{array}{c} \times \\ \times \text{---} s \text{---} \times \\ \times \end{array}$$

$$\longrightarrow [H, W(\gamma)] = 0 \quad \forall \quad \gamma \in \mathbb{Z}_1(\gamma; \mathbb{Z}_2)$$

→ Exactly-solvable: $\{A_s, B_p\}$ are mutually commuting.

⇒ Ground state space: $A_s = B_p = \mathbb{1}$.

⇒ $\mathcal{H}_{gs} \subset \mathcal{H}_{top}$: $\mathbb{Z}_2$ 1-form symmetry arising from $\{W(\gamma)\}_{\gamma \in \mathbb{Z}_2, (1; \mathbb{Z}_2)}$

→ can insert symmetry defect by truncating $W(\gamma)$:

$$H \xrightarrow[\text{@ site } r]{W \text{ defect}} H_r = H + 2A_r.$$

→ On full Hilbert space,

⇒ topological defect:

$$H_{r_1} = U_{12} H_{r_2} U_{12}^\dagger \quad \text{with unitary } U_{12} = \prod_{i \rightarrow j} Z.$$

Outlook of these lectures

We talked about

- 1) Unitary symmetry operators, their defects, and anomalies on the lattice
- 2) Non-invertible and higher-form symmetry on the lattice

Didn't have time for a lot of cool topics!

operator algebra framework, onsiteability, blendability, Families of lattice systems, phases of matter, gauging, fermionic lattice systems, more general crystalline symmetries and their LSM anomalies, subsystem symmetry and fractons, Anomalies of continuous unitary symmetry and generalized symmetry, Lattice higher-group symmetry, Ancillae, ...

Lots of exciting open directions!

In my view, the most exciting frontier of symmetry is symmetry on the lattice

- Use symmetry to clarify relationship between quantum lattice systems and QFTs.
- Use symmetry to characterize, classify, and constrain quantum entanglement and quantum phases of matter

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