

Lecture 2

Last lecture

- Lattice Hamiltonian Systems
- Unitary Symmetry operators: onsite $\subset$ FQCS $\subset$ QCA's

This lecture: defects and anomalies

Defects

A defect is a localized modification to a theory.

In a Hamiltonian theory, a defect D with support Σ modifies: ↗ in space

1) The Hilbert Space

$$\mathcal{H}_r = \bigotimes_{i \in r} \mathcal{H}_i \longmapsto \mathcal{H}_r^{(D, \Sigma)}$$
$$\rightarrow \mathcal{H}_r^{(D, \Sigma)} = \mathcal{H}_r \quad \text{if} \quad r \cap \Sigma = \emptyset.$$

↗ called defect Hilbert Space

2) Hamiltonian

$$H \longmapsto H_0^{(\Sigma)}$$

↗ called defect Hamiltonian.

$\rightarrow H_0^{(\Sigma)}$ acts like H away from Σ .

$\rightarrow$ If every $\mathcal{H}_r^{(D, \Sigma)} = \mathcal{H}_r$:

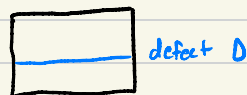
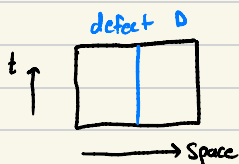
$$H_0^{(\Sigma)} = H + \delta H_0^{(\Sigma)}$$

with $\text{Supp}(\delta H_0^{(\Sigma)}) \subseteq \Sigma^{+w}$

$\rightarrow$ Other operators also modified within Σ^{+w}

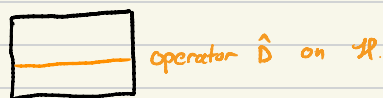
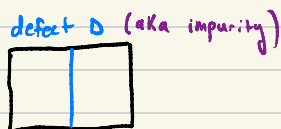
Spacetime pov comment

Euclidean theory: (pov of Iñaki's lecture)



→ Both are localized modifications of Euclidean path integral

Hamiltonian interpretation



Partition function:

$$\text{Tr}_{\mathcal{H}^{(0,z)}} [e^{-\beta H_0^{(z)}}]$$

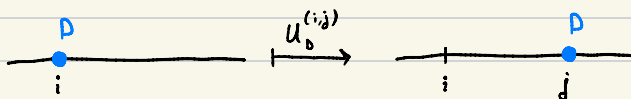
$$\text{Tr}_{\mathcal{H}} [\hat{D} e^{-\beta H}]$$

Symmetry defects

A symmetry defect is

- 1) a topological defect: a defect whose support is changed using unitary "movement" operators

e.g., 1d (picture in space, not spacetime)



Unitary $U_0^{(i,j)}$ has support $\text{Supp}(U_0^{(i,j)}) \subset [i, j]^{+w}$,

$$\Rightarrow H_0^{(j)} = U_0^{(i,j)} H_0^{(i)} U_0^{(i,j)\dagger}$$

$\Rightarrow$ Energy spectrum independent of Symmetry defect's location

2) Implements a Symmetry transformation across space:

$$U_D^{(i,j)} O_K U_D^{(i,j)\dagger} = \begin{cases} \hat{D} O_K \hat{D}^\dagger & i \ll K \ll j \\ O_K & K \ll i \text{ or } K \gg j \end{cases}$$

$\Rightarrow D$ is the Symmetry defect for the Symmetry operator $\hat{D}$.

$$\Rightarrow \hat{D} = \prod_{j=1}^L U_D^{(j,j+1)} \quad (\text{defect} \rightarrow \text{op})$$

op $\rightarrow$ defect { For an FDQC U , a U -Symmetry defect can be inserted using a locality-preserving unitary U_B satisfying within B .

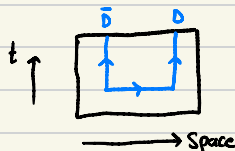
$$U_B O U_B^\dagger = \begin{cases} U O U^\dagger & \text{Supp}(O) \text{ deep inside of } B \\ 0 & \text{Supp}(O) \text{ far outside of } B \end{cases}$$

(U_B is a truncation of U to B)

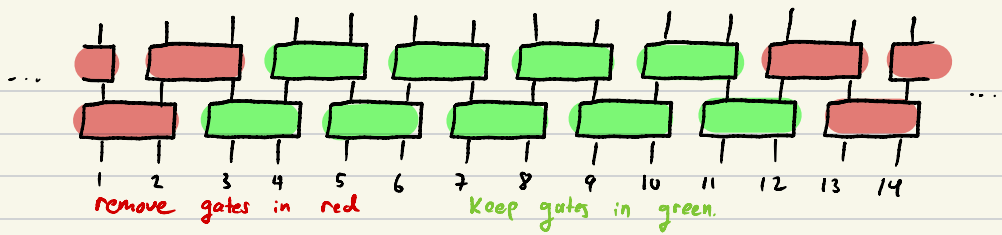
$\rightarrow U_B$ inserts a U -Symmetry defect at ∂B .

In 1+1d

$\rightarrow$ Spacetime intuition: truncated operator corresponds to Euclidean theory defect



$\rightarrow$ Example with depth 2 FDQC.



$\Rightarrow$ Truncation of the FDQC to the interval $[3, 12]$

$\rightarrow$ Fails for a QCA that is not an FDQC: A nontrivial GNVW index is an obstruction to a locality-preserving, unitary truncation.

(Because such QCAs transport information across space.)

Ising chain example

Ising chain with a η -Symmetry defect

1) Truncate η to interval $I = [j_L, j_R]$

$$\eta_I = \prod_{j=j_L}^{j_R} X_j$$

2) Conjugate H :

$$\eta_I H \eta_I^\dagger = H + 2 Z_{j_R-1} Z_{j_R} + 2 Z_{j_R} Z_{j_R+1}$$

3) Defect Hamiltonian: $H_\gamma^{(j_R)} = H + 2 Z_{j_R} Z_{j_R+1}$

$\rightarrow$ Movement operator $M_{j_R} = \eta_{[j_L, j_R+1]} \eta_I^\dagger = X_{j_R+1}$

$$\Rightarrow M_{j_R} H_\gamma^{(j_R)} M_{j_R}^\dagger = H_\gamma^{(j_R+1)}$$

Twisted translations:

$$\gamma_{\pm} T \gamma_{\pm}^{\dagger} = X_{j_L} X_{j_{R+1}} T \Rightarrow T_{\gamma}^{(in)} = X_{j_{R+1}} T$$

$$\rightarrow \text{Satisfies } (T_{\gamma}^{(in)})^L = \gamma.$$

Anomalies

Consider discrete G -symmetry $\{U_g\}_{g \in G}$ in 1+1d with

1) control space $X = S^1$

2) every U_g is a FDQC $\rightarrow$ automatically true for finite G with GCA symmetry operators

Truncate each U_g to U_g^I with $I = [j_L, j_R]$ and sites j_L and j_R far away.

$$U_g^I U_h^I = \Omega_{g,h} U_{gh}^I.$$

$\rightarrow$ An operator.

$\rightarrow$ Separation of j_L and j_R

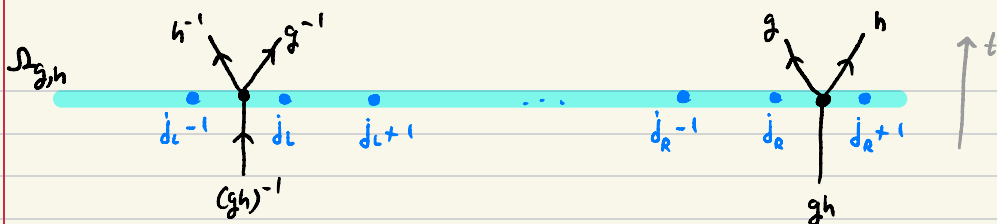
$$\Omega_{g,h} = \Omega_{g,h}^L \Omega_{g,h}^R$$

with $\Omega_{g,h}^{L/R}$ localized around $j_{L/R}$.

$\rightarrow$ assume $U_1^I = \mathbb{1} \Rightarrow \Omega_{1,g} = \Omega_{g,1} = \mathbb{1}$.

U_g^I creates a g -defect near j_R and g^{-1} defect near j_L .

$$\rightarrow \Omega_{g,h} = U_g^I U_h^I (U_{gh}^I)^{\dagger}:$$



$\Rightarrow \Omega_{g,h}^R$ is a "splitting" operator for a gh defect.

Associativity: $(U_g^I U_h^I) U_k^I = U_g^I (U_h^I U_k^I)$

$\Rightarrow \Omega_{g,h} \Omega_{gh,k} = (U_g^I \Omega_{h,k} (U_g^I)^+) \Omega_{g,hk}$

$\Rightarrow \Omega_{g,h}^R \Omega_{gh,k}^R = \omega(g,h,k) (U_g^I \Omega_{h,k}^R (U_g^I)^+) \Omega_{g,hk}^R$

Where $\omega: G^3 \rightarrow U(1)$

Graphically

$$= \omega(g,h,k)$$

The anomaly of the G -symmetry is classified by the cohomology class

$$[\omega] \in \mathcal{H}^3(G, U(1))$$

$\rightarrow$ From its def., ω satisfies the 3-cocycle condition

→ Redefining $\Omega_{g,h}^R \mapsto e^{i\theta(g,h)} \Omega_{g,h}^R$ changes ω by a 3-coboundary

Note: Agrees with the classification of 't Hooft anomalies in QFT.

(Suprising: The lattice is non-relativistic. No continuum limit needed)

Example: CZX Symmetry

Consider 1d lattice system of even L sites with a qubit per site, and the FQCC

$$U = \prod_{j=1}^L CZ_{j,j+1} \prod_{j=1}^L X_j$$

→ Controlled-Z gate $CZ_{j,k} = \frac{1}{2}(1 + Z_j + Z_k - Z_j Z_k)$

$$(CZ_{j,k} X_j CZ_{j,k}^\dagger = X_j Z_k, \quad CZ_{j,k} X_k CZ_{j,k}^\dagger = X_k Z_j)$$

U satisfies:

$$U X_j U^\dagger = Z_{j-1} X_j Z_{j+1}, \quad U Z_j U^\dagger = -Z_j$$

$\Rightarrow U^2 = 1 \Rightarrow U$ generates a $\mathbb{Z}_2$ symmetry

A symmetric Hamiltonian:

$$H = \frac{1}{2} \sum_{j=1}^L (X_j + Z_{j-1} X_j Z_{j+1})$$

Fun facts: 1) IR limit is $R=\sqrt{2}$ compact boson CFT

2) critical point between inequivalent $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT phases

U-Symmetry defect

1) Truncate U to interval $I = [j_L, j_R]$

$$U_I = \prod_{j=j_L}^{j_R-1} C Z_{j, j+1} \prod_{j=j_L}^{j_R} X_j$$

2) Conjugate H :

$$U_I H U_I^\dagger = H + \delta H_L + \delta H_R$$

with, e.g.,

$$\delta H_R = -X_{j_R} Z_{j_R, j_R+1} C Z_{j_R-1, j_R+1} - Z_{j_R} X_{j_R+1} Z_{j_R+2}$$

3) Defect Hamiltonian: $H_U^{(j_R)} = H + \delta H_R$.

→ Movement operator

$$M_{j_R} = U_{[j_L, j_R+1]} U_I^\dagger = C Z_{j_R, j_R+1} X_{j_R+1}$$

Anomaly of $\mathbb{Z}_2$ Symmetry $\{U^n\}_{n \in \mathbb{Z}_2}$.

$$U_I^2 = (-1)^{|I|-1} Z_{j_L} Z_{j_R} \Rightarrow \Omega_{m,n} = \left((-1)^{|I|-1} Z_{j_L} Z_{j_R} \right)^{mn}$$

$$\Rightarrow \Omega_{m,n}^R = Z_{j_R}^{mn}$$

→ anomaly 3-cocycle

$$\omega(m,n,k) = \Omega_{m,n}^R \Omega_{m+n,k}^R \left(U_m^I \Omega_{n,k}^R (U_m^I)^\dagger \Omega_{m,n+k}^R \right)^\dagger$$

$$= (-1)^{mnK}$$

$\Rightarrow$ The cohomology class of $[W]$ is nontrivial.
 Since every normalized 3-coboundary satisfies $\delta\mu(1,1,1) = 1$.

$\Rightarrow U$ generates an anomalous $\mathbb{Z}_2$ symmetry.

(becomes the diagonal $\mathbb{Z}_2^M \times \mathbb{Z}_2^N$ sym in the continuum)

Remarks:

1) The anomaly of a symmetry is not a feature of a specific Hamiltonian

2) Onsite symmetries are anomaly-free

$\rightarrow$ If every $U_g = \prod_i U_i^{g_i} \Rightarrow$ every $\Omega_{g,h,k} = 1$.

3) For every GCA W , the anomaly is invariant under

$$U_g \mapsto W U_g W^\dagger \equiv \tilde{U}_g$$

$\rightarrow$ Can use $\tilde{U}_g^I = W U_g^I W^\dagger \Rightarrow \tilde{\Omega}_{g,h}^R = W \Omega_{g,h}^R W^\dagger$ which has same 3-cycle ω as $\Omega_{g,h}^R$.

4) The anomaly of $\{U_g \otimes \tilde{U}_g\}_{g \in G}$ is the product $[W][\tilde{W}]$

5) $\exists$ a 2+1d generalization (Kapustin, Xu '25 ; Kawagoe, Shirley '25)

$\rightarrow$ classified by $\mathcal{H}^*(G, U(1))$

This 1+1d anomaly on the lattice implies:

- 1) Obstruction to gauging (Seifnashri 2023)
- 2) Obstruction to trivially gapped phase (Kapustin & Sopenko 2024)
- 3) Anomaly inflow with a 2+1d SPT
(Elie & NayaK 2014, Kawagoe & Levin 2021)

General QCDs

Suppose discrete G -symmetry $\{U_g\}_{g \in G}$ with each U_g a QCD but not necessarily an FDQC

→ ie, U_g could be lattice translations.

The anomaly can be calculated using the following trick.
(Kapustin & Sopenko 2024)

- 1) Double the system: $\mathcal{H}_j \mapsto \mathcal{H}_j \otimes \mathcal{H}_j$
- 2) Consider symmetry operators

$$U_g \otimes \mathcal{U}_g^{-1}$$

For prime $\dim \mathcal{H}_j = p$, each $U_g = T^{n(g)} C_g$. Let

$$\mathcal{U}_g = T^{n(g)}$$

(Generalizer based on prime factors of $\dim \mathcal{H}_j$)

⇒ From GNVW index:

$$\mathcal{U}_g \mathcal{U}_h = \mathcal{U}_{gh}, \quad U_g \otimes \mathcal{U}_g^{-1} \text{ is an FDQC}$$

3) The $\{\gamma_g\}_{g \in G}$ symmetry is manifestly anomaly free

$\Rightarrow$ Anomaly of $\{U_g \otimes \gamma_g^{-1}\}_{g \in G}$ is the same as that of $\{U_g\}_{g \in G}$.

Example: LSM anomaly

Infinite lattice with $\mathbb{Z} \times G$ Symmetry operators:

1) Lattice translation T (not an FDQC)

2) Onsite G -symmetry operators $U_g = \prod_i U_g^{(i)}$

$\rightarrow U_g^{(i)} U_h^{(j)} = v(g, h) U_{gh}^{(j)}$ with 2-cocycle $v: G^2 \rightarrow U(1)$

$\rightarrow$ Each unit-cell associated with projective representation $[v]$.

Stacking trick:

$$T \mapsto T \otimes T^{-1}$$

$$U_g \mapsto U_g \otimes \mathbb{1}.$$

Can show that the anomaly cocycle is

$$\omega((n_1, g_1), (n_2, g_2), (n_3, g_3)) = v(g_1, g_2)^{n_3}$$

$$\Rightarrow [\omega] \neq 1 \quad \text{iff} \quad [v] \neq 1$$

$\Rightarrow$ Related to $\mathcal{H}^1(\mathbb{Z}, \mathcal{H}^2(G, U(1))) \cong \mathcal{H}^2(G, U(1))$ in the Künneth formula of $\mathcal{H}^3(\mathbb{Z} \times G, U(1))$

$\Rightarrow$ Finite-system method: Projective representation of $\{u_g\}_{g \in G}$ at L sites differs from that at $L-1$ sites by $[U]$.

Called a Lieb-Schultz-Mattis (LSM) anomaly

$\rightarrow$ Generalization of the LSM theorem.