

Symmetry in Quantum Lattice Systems.

Lecture plan

1) Unitary Symmetry operators on the lattice

Nachtergaele, Sims, Young '18 ; Freedman, Haah, Hastings '19 ; Kopustin, Sopenko '24

2) Symmetry defects and anomalies on the lattice

Else, Nagak '14 ; Cheng, Seiberg '22 ; Seifnashri '23 ; Kopustin, Sopenko '24

3) Generalized Symmetry on the lattice,

Qi, Rudzihovskiy, Hermele '20 ; Seiberg, Shao '23 ; Okada, Tachikawa '24 ;
SP, Lam, AKroy '25

Goal: Introduce basic aspects of the framework with examples
(less emphasis on applications due to time.)

Why care about the lattice?

1) QFT pov: Mathematically controlled setting for non-perturbative physics.
Computational study and quantum-simulation

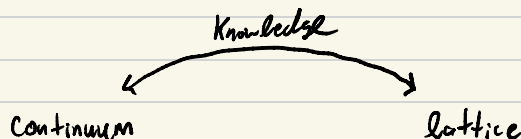
2) Quantum lattice matter: Condensed matter and AMO systems.

Two slogans:

1) Sometimes the lattice is a regulator. Sometimes it's the physics.

2) Sometimes the QFT is the starting point. Sometimes it's the answer.

Mutualistic relationship



→ Modern examples

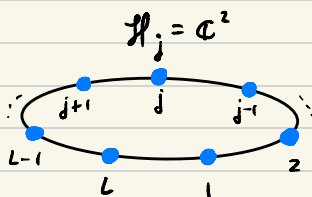
Foliated field theory ← ————— Fractons

Generalized Symmetry —————→ New lattice models
with interesting phases and phase transitions

Hamiltonian lattice system: Spatial lattice and continuous time.

Example: Ising Chain (Warm up example)

1d lattice system on a ring with a qubit on each site j and total # of sites = L → aka spin 1/2.



Onsite tensor product Hilbert Space

$$\mathcal{H} = \bigotimes_{j=1}^L \mathbb{C}^2$$

→ site j Pauli operators X_j and Z_j ($j \sim j+L$):

$$X_j^2 = Z_j^2 = 1 \quad X_j Z_k = (-1)^{\delta_{j,k}} Z_k X_j$$

$$\text{Hamiltonian } H = - \sum_{j=1}^L Z_j Z_{j+1} - h \sum_{j=1}^L X_j \quad (h \in \mathbb{R})$$

→ Locality: Each term involves at most neighboring qubits

General Set up

A $(n+1)d$ lattice system is Specified by:

→ or other dimensional cells

- 1) Spatial lattice: Discrete Set of lattice sites Λ with an n -dimensional metric Space X and locally finite, injective map

$$\Lambda \rightarrow X$$

→ X gives a notion of spatial dimension and distance

- 2) Onsite Hilbert spaces $\{\mathcal{H}_j\}_{j \in \Lambda}$

Hilbert space in finite region Γ : $\mathcal{H}_\Gamma = \bigotimes_{j \in \Gamma} \mathcal{H}_j$.

→ For finite Λ , total Hilbert space is $\mathcal{H}_\Lambda$.

→ For infinite Λ , $\mathcal{H}_\Lambda$ is ill-defined.

e.g., For $|\psi\rangle = \bigotimes_{j \in \Lambda} |\psi_j\rangle$ and $|\phi\rangle = \bigotimes_{j \in \Lambda} |\phi_j\rangle$, the inner-product $\langle \psi | \phi \rangle = \prod_{j \in \Lambda} \langle \psi_j | \phi_j \rangle$ doesn't converge.

(Total Hilbert space found using a reference state: GNS construction)

- 3) Local projectors $\{P_j\}_{j \in \Lambda}$

→ A local operator O is physical iff $O P_j = P_j O$

- 4) Hamiltonian $H = \sum_{r \in \Lambda} h_r \otimes \mathbb{1}_{\Lambda \setminus r}$

→ Each h_r a Hermitian operator of $\mathcal{H}_r$.

→ Strict-locality: $h_r = 0$ if $\text{diam}(\Gamma) > \Gamma_H$ with Γ_H the finite range of H .

→ Independent of $|\Lambda|$ for finite Λ .

→ For infinite Λ , only $[O, H] = i\partial_t O$ is a well defined operator.

$$[O, H] = \sum_{n: n \cap \text{supp}(O) \neq \emptyset} [O, h_n \otimes \mathbb{1}_{\Lambda \setminus n}] \quad \left. \vphantom{\sum} \right\} \text{A finite sum.}$$

Comments:

1) locality baked into the Set up.

2) Infinite Λ is subtle (we'll try to stick to finite Λ)

We will assume: (For simplicity)

1) $\dim \mathcal{H}_i < \infty$

2) Lectures 1 & 2: No nontrivial projectors P_j .

3) Lectures 1 & 2 in 1+1d, Lecture 3 in 1+1d and 2+1d

Grouplike Symmetry

An operator U is a symmetry operator iff U is

1) (anti-)Unitary: $U^\dagger = U^{-1}$ (we'll restrict to unitaries)

Not needed in 0+1d QM. $\left\{ \begin{array}{l} 2) \text{locality-respecting: } U \circ U^\dagger \text{ is a local operator for each local operator } O \end{array} \right.$

→ a group
A G -Symmetry is a collection $\{U_g\}_{g \in G}$ of symmetry operators satisfying

1) Group law: $U_g U_h = e^{i\Theta(g,h)} U_{gh}$.

2) Faithfulness: $U_g \propto \mathbb{1}$ iff $g=1$.

The Symmetry of a Hamiltonian H is a G -Symmetry $\{U_g\}_{g \in G}$ such that

$$[U_g, H] = 0 \quad \forall g$$

→ Note: Symmetry can be stated without a reference symmetric Hamiltonian in mind.

Ising Chain Symmetry:

Consider the operators

$\eta = \prod_{j=1}^L X_j$	$\eta X_j \eta^\dagger = X_j$	$\eta Z_j \eta^\dagger = -Z_j$
translation operator T	$T X_j T^\dagger = X_{j+1}$	$T Z_j T^\dagger = Z_{j+1}$
reflection operator R	$R X_j R^\dagger = X_{-j}$	$R Z_j R^\dagger = Z_{-j}$

→ All are Symmetry operators

→ Satisfy

$$\eta^2 = 1$$

$$T^L = 1$$

$$R^2 = 1$$

$$\eta T = T \eta$$

$$\eta R = R \eta$$

$$R T = T^{-1} R$$

⇒ Generate a $\mathbb{Z}_2 \times D_{2L}$ Symmetry

→ Satisfy $[\eta, H] = [T, H] = [R, H] = 0$

⇒ $\mathbb{Z}_2 \times D_{2L}$ Symmetry of the Ising Chain.

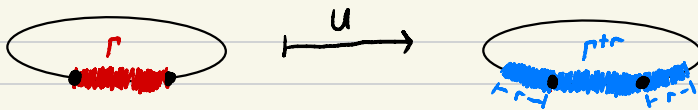
Quantum Cellular Automaton (QCA)

A special class of Unitary Symmetry operators are QCAs.

→ U is a QCA if for each local operator o supported in Γ ,

$$\text{Supp}(U o U^\dagger) \subseteq \Gamma^{+r}$$

With Γ^{+r} all sites within a finite distance r of Γ .



⇒ Minimal such r is called the range of U .

→ QCAs generate strictly locality-preserving symmetries.

In the Ising chain:

Symmetry operator	QCA?
$\gamma = \prod_{j=1}^L X_j$	Yes (range 0)
translation T	Yes (range 1)
reflection R	No

A QCA U with range 0 is called an onsite Unitary

→ e.g., γ is onsite.

$\rightarrow U$ is onsite $\Leftrightarrow U = \bigotimes_{j \in \Lambda} U_j$ with $U_j \in \mathcal{U}(\mathcal{H}_j)$

Finite-depth quantum circuit (FDQC)

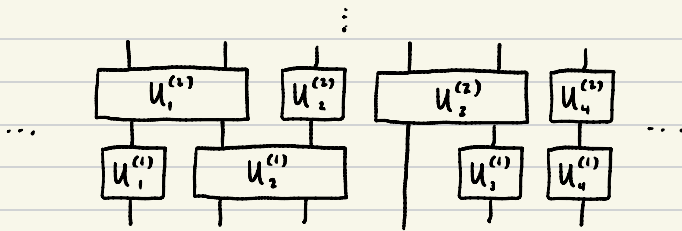
A QCA U is a FDQC with depth D iff ↗ finite in thermodynamic limit.

$$U = U^{(D)} U^{(D-1)} \dots U^{(1)}, \quad U^{(k)} = \prod_a U_a^{(k)}$$

with $\{U_a^{(k)}\}$ local unitary operators satisfying

$$\text{Supp}(U_a^{(k)}) \cap \text{Supp}(U_b^{(k)}) = \emptyset \quad \forall a \neq b.$$

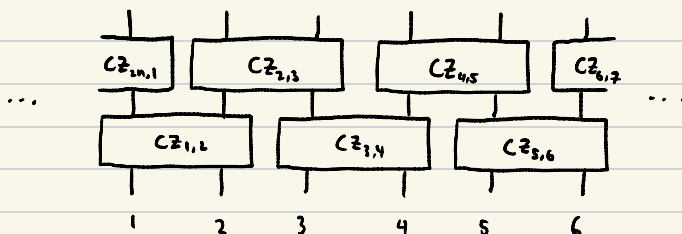
$\rightarrow U_a^{(k)}$ are called gates.



$$\{\text{Onsite Unitaries}\} \subset \{\text{FDQCs}\} \subset \{\text{QCAs}\}$$

with $CZ_{j,k} = \frac{1}{2}(1 + z_j z_k - z_j z_k)$

Example: $\prod_{j=1}^{2n} CZ_{j,j+1} = \prod_{\text{even } j} CZ_{j,j+1} \prod_{\text{odd } j} CZ_{j,j+1}$



} Brick-wall circuit.

Anti-example: Lattice translation T

[Gross - Neeme - Vogts - Werner (GNVW) 2009]

GNVW index

Every QCA U has an index $\text{ind}(U)$

→ Measures net transport quantum information across a spatial cut.

$$\Rightarrow \text{For translation } T O_j T^\dagger = O_{j+1}$$

$$\text{ind}(T) = \dim \mathcal{H}_j.$$

For bosonic 1+1d lattice systems

(Won't give full def here)

$$\text{ind}(U) = \text{GNVW index of } U \in \mathcal{Q}_{\geq 0}$$

Group homomorphism $\text{ind}: G_{\text{QCA}} \rightarrow \mathcal{Q}_{\geq 0}$

$$\text{ind}(UV) = \text{ind}(U)\text{ind}(V) \quad \text{ind}(U^{-1}) = \text{ind}(U)^{-1}$$

→ Satisfies

$$\text{ind}(U) = 1 \iff U \text{ is an FDQC} \quad \text{ind}(U \otimes V) = \text{ind}(U)\text{ind}(V)$$

⇒ T is not an FDQC

Examples:

$$1) \text{ qubit translations: } \text{ind}(T_{\text{qubit}}) = 2, \quad \text{ind}(T_{\text{qubit}}^{-1}) = 1/2$$

$$2) \text{ qutrit translations: } \text{ind}(T_{\text{qutrit}}) = 3, \quad \text{ind}(T_{\text{qutrit}}^{-1}) = 1/3$$

$$3) \mathcal{H}_j = \mathbb{C}^2 \otimes \mathbb{C}^3: \quad \text{ind}(T_{\text{qubit}} T_{\text{qutrit}}) = 6, \quad \text{ind}(T_{\text{qubit}} T_{\text{qutrit}}^{-1}) = 2/3$$

4) For prime $\dim \mathcal{H}_j = p$ Every QCA U admits decomposition $U = T^n C$ with C an FDQC

$$\text{ind}(U) = \text{ind}(T)^n \text{ind}(C) = p^n.$$

Remarks:

1) For every QCA U , $U \otimes U^{-1}$ is an FDQC

$$\rightarrow \text{ind}(U \otimes U^{-1}) = \text{ind}(U) \text{ind}(U)^{-1} = 1$$

$\rightarrow$ Explicit expression:

let $U_A = U \otimes \mathbb{1}$, $U_B = \mathbb{1} \otimes U$, and $S = \prod_j S_j$ swaps the two tensor systems.

$$\Rightarrow U \otimes U^{-1} = U_A U_B^{-1} = U_A S U_A^{-1} S = \prod_j U_A S_j U_A^{-1} \prod_j S_j$$

$\rightarrow$ in thermodynamic limit

2) For a finite G -symmetry of QCA, $\{U_g\}_{g \in G}$, each U_g is an FDQC.

$\rightarrow$ Every U_g satisfies $U_g^{\text{ord}(g)} \propto \mathbb{1}$.

$$\Rightarrow \text{ind}(U_g^{\text{ord}(g)}) = 1$$

$$\Rightarrow \text{ind}(U_g) = 1$$