

[Euler-Lagrange equations]

Def: A symmetry in classical mechanics is a transformation of the fields $\phi \rightarrow \phi'$ such that the action stays invariant: $S[\phi] \rightarrow S[\phi'] = S[\phi]$. Assume, for simplicity, that $\mathcal{L}[\phi] \rightarrow \mathcal{L}[\phi'] = \mathcal{L}[\phi]$ (so no boundary terms)

Example

$$S = \int_{\mathbb{R}^d} d^d x \mathcal{L} = \int_{\mathbb{R}^d} d^d x (\partial_\mu \bar{\phi} \partial^\mu \phi + m \bar{\phi} \phi)$$

with $\phi: \mathbb{R}^d \rightarrow \mathbb{C}$ a complex scalar. The symmetry group in this case is $U(1)$: $\phi \rightarrow \phi^{(\alpha)} := \phi e^{i\alpha}$ with $\alpha \in \mathbb{R}/2\pi$.

Theorem (Noether) Whenever we have a transformation

$\phi_i \rightarrow \phi_i + \alpha a_i + O(\alpha^2)$ with α constant such that

$$\mathcal{L}(\vec{\phi}, \partial \vec{\phi}) \rightarrow \mathcal{L}(\vec{\phi}, \partial \vec{\phi}) + O(\alpha^2) \quad [\text{i.e. a symmetry}]$$

the Noether current $j^\mu = \sum_i a_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}$ is conserved:

$$\partial_\mu j^\mu = 0.$$

without using the equations of motion!

Proof (for a single field, the generalisation is straightforward):

$$\mathcal{L}(\phi, \partial_\mu \phi) \rightarrow \mathcal{L}(\phi^{(\alpha)}, \partial_\mu \phi^{(\alpha)}) \quad \text{with } \phi^{(\alpha)} := \phi + \alpha a + O(\alpha^2)$$

$$\begin{aligned} &= \mathcal{L}(\phi, \partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) + \dots \quad (\delta \phi := \alpha a) \\ \stackrel{E-L}{=} &= \mathcal{L}(\phi, \partial_\mu \phi) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta \phi + \dots \end{aligned}$$

$$= \mathcal{L}(\phi, \partial_\mu \phi) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi \right) + \dots$$

$$= \mathcal{L}(\phi, \partial_\mu \phi) + \alpha \underbrace{\partial_\mu \left(a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}_{j^\mu} + (\partial_\mu \alpha) \underbrace{\left(a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}_{j^\mu} + \dots$$

But by assumption

$$\mathcal{L}[\phi] = \mathcal{L}[\phi']$$

when $\partial_\mu \alpha = 0$ so the $\partial_\mu j^\mu$ term needs to vanish. $\square$.

Note that this contribution appears even if we don't apply the Euler-Lagrange equations! (useful later)

Example In the case of the charged complex boson we have $a = \frac{d(e^{iq\alpha}\phi)}{d\alpha} \Big|_{\alpha=0} = iq\phi$ and $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \bar{\phi}$, so

$$j^\mu = iq\phi \partial^\mu \bar{\phi} - iq\bar{\phi} \partial^\mu \phi = 2q \operatorname{Im}(\bar{\phi} \partial^\mu \phi) \quad (\text{Note: this is real})$$

[Exercise: show that $\partial_\mu j^\mu = 0$ from E-L equations]

Out of the current we can construct the Noether charge

$$Q = \int_{\mathbb{R}^3} d^3x j_0$$

It has the property that it is conserved in time (given appropriate boundary conditions):

$$\frac{dQ}{dt} = \int_{\mathbb{R}^3} d^3x \partial_0 j_0 \stackrel{\partial_\mu j^\mu = 0}{=} \int_{\mathbb{R}^3} d^3x \vec{\nabla} \cdot \vec{j} = \int_{S_\infty^2} \vec{j} \cdot \hat{n} d\Omega_{S^2} = 0$$

(j₁, j₂, j₃)

In the Hamiltonian formalism the action of the symmetry is given by taking the Poisson bracket with the conserved charge:

$$\{\phi, Q\} = iq\phi \quad [\text{Exercise}]$$

or exponentiating:

$$e^{i\alpha\{ \cdot, Q \}} \phi = e^{iq\alpha} \phi$$

Grönewald's theorem

In the quantum theory, as usual, we have

$$[\hat{f}, \hat{g}] = i \{f, g\} \quad (\text{up to higher order terms})$$

so in particular for an operator of charge q under $\hat{Q}$ we have

$$[\hat{Q}, \hat{O}_q] = q \hat{O}_q$$

The quantum operator $\hat{Q}$ is hermitian, so we can construct a unitary operator $\hat{U}_\alpha$ as $\hat{U}_\alpha := \exp(i\alpha \hat{Q})$. The action of this operator on $\hat{O}_\alpha$ is:

$$\hat{U}_\alpha \hat{O}_\alpha \hat{U}_\alpha^\dagger = e^{i\alpha\beta} \hat{O}_\alpha \quad \text{[Exercise: show this]}$$

A different way of understanding this action on $\hat{O}_\alpha$ is simply to notice that acting on any state $|\psi\rangle$:

$$|\psi\rangle \rightarrow \hat{U}_\alpha |\psi\rangle$$

so the action above is required by consistency:

$$\hat{O}_\alpha |\psi\rangle \rightarrow \hat{U}_\alpha \hat{O}_\alpha |\psi\rangle = \hat{U}_\alpha \hat{O}_\alpha \hat{U}_\alpha^\dagger \hat{U}_\alpha |\psi\rangle$$

Notes: (1) Henceforth I will drop hats.

(2) Clearly $U_\alpha U_\beta = e^{i\alpha Q} e^{i\beta Q} = e^{i(\alpha+\beta)Q} = U_{\alpha+\beta}$,

so these operators form an abelian group.

The story generalises straightforwardly to any group G :

for $g, h \in G$ we can define U_g, U_h and $U_g U_h = U_{gh}$.

$$\underset{\substack{\text{rep} \\ R(g)}}{R(g)} = \exp(i \sum_k \underset{\substack{\text{algebra} \\ \text{generators}}}{g_k} t_k^R) \Rightarrow U_g = \exp(i \sum_k \underset{\substack{\text{charges, one for each} \\ \text{Lie algebra generator}}}{g_k} Q_k^R)$$

The action of these generators on operators in some rep R of G is:

$$U_g O_R^i U_g^\dagger = R(g)_j^i O_R^j$$

Despite writing it for continuous groups, the final expressions make sense for discrete groups too, such as $\mathbb{Z}_n$ or S_k .

In path integral language, we can translate the statement

$$U_g \bigcirc U_g^\dagger = R(g) \bigcirc$$

to
Operator statement, so we don't fix the boundaries.

interpretation: gluing after acting with $U_g^\dagger$ on $\mathcal{H}$

where we have slightly separated the operators in the time direction for clarity.

Note that symmetries commute with the Hamiltonian

This is broken if we insert a charged operator in the time evolution cylinder:

The difference is measured precisely by the formula above.

Note that in the Euclidean formulation time is on equal footing with the other directions, so we generalise invariance under

time translations (commuting with H) with invariance under continuous

deformations (commuting with $T_{\mu\nu}$). The generalisation of $\int_{\mathbb{R}^{d-1}} j^0 d^{d-1}x$

is:

$$Q(\Sigma^{d-1}) := \int_{\Sigma^{d-1}} j_\mu n^\mu d\Sigma$$

Going back to our action on operators, we have

$$\begin{array}{c} \text{---} U_g \\ \cdot G \\ \text{---} U_g^+ \end{array} \xrightarrow{\text{deform}} \begin{array}{c} U_g \\ \cdot G \\ U_g^+ \end{array} \quad \text{with } U_g U_g^+ = 1 \text{ and } U_g^+ U_g = 1$$

So we conclude:

$$R(g) \mathcal{O}'' \quad (\text{topological operator})$$

This was the basic starting point in the Generalised Global Symmetries paper [Gaiotto, Kapustin, Seiberg, Willett '14].

— PROOF OF THE WARD IDENTITY —

Proof: consider a transformation $\phi \rightarrow \phi + \alpha(x)a$. By the argument given above (for $\partial_\mu j^\mu = 0$) you find that if the transformation is a symmetry for α constant, then $S' \rightarrow S - i \int d\alpha j$

Notes:
• Diff notation
• Convenient
d of j !

Assuming $\partial_\mu j^\mu = 0$ (or more generally, $\alpha|_{\partial M^d} = 0$) we have

$$0 = \int_{\partial M^d} \alpha j = \int_{M^d} d(\alpha j) = \int_{M^d} (d\alpha) \wedge j + \int_{M^d} \alpha \wedge dj$$

$$S \rightarrow S + i \int \alpha \wedge dj$$

For 0-forms we generally leave the wedge product implicit:

$$S \rightarrow S + i \int \alpha dj$$

Consider the path integral

$$I = \int D\phi \, O_1(y_1) \cdots O_n(y_n) e^{-S[\phi]}$$

Changing variables to $\phi' = \phi + \alpha a$ we have (by assumption $0 \Rightarrow 0 + \alpha \delta 0 + \dots$)

$$I = \int D\phi' (O_1(y_1) + \alpha \delta O_1(y_1)) \cdots (O_n(y_n) + \alpha \delta O_n(y_n)) e^{-S[\phi']} - i \int \alpha dj$$

If the symmetry is not anomalous, we have $D\phi' = D\phi$. Expanding to first order in α :

$$\begin{aligned} I &= \int D\phi \, O_1(y_1) \cdots O_n(y_n) e^{-S} \\ &+ \int D\phi \sum_{i=1}^n O_1(y_1) \cdots \alpha(y_i) \delta O_i(y_i) \cdots O_n(y_n) e^{-S} \\ &- i \int D\phi \int \alpha (dj) O_1(y_1) \cdots O_n(y_n) e^{-S} + O(\alpha^2) \end{aligned}$$

$$\text{writing } \alpha(y_i) \delta O_i(y_i) = \int_M \delta^{(4)}(y_i) \alpha \delta O_i$$

and recalling that the terms proportional to α need to

vanish for arbitrary α :

$$\text{idj } \mathcal{O}_1(y_1) \cdots \mathcal{O}_n(y_n) = \sum_{i=1}^n \delta^{(4)}(y_i) \mathcal{O}_1(y_1) \cdots [\delta \mathcal{O}_i(y_i)] \cdots \mathcal{O}_n(y_n)$$

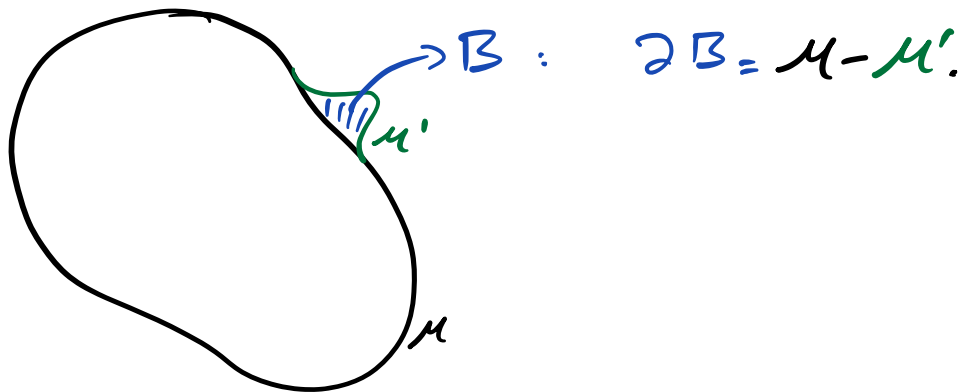
with $\delta^{(4)}(y) := \delta(x-y_0) \delta(x_1) \delta(x_2) \delta(x_3) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$
a point in $\mathcal{M}^4$

or more abstractly:

$$\int_{\mathcal{M}} \rho \delta^{(4)}(y) = \begin{cases} \rho(y) & \text{if } y \in \mathcal{M} \\ 0 & \text{if } y \notin \mathcal{M} \end{cases}$$

What the Ward identity means is that as long as we don't encounter charged operators we can deform the symmetry generator $U_\alpha = \exp(i\alpha \int_{\Sigma} j^\mu_\nu d\Sigma^\nu)$ freely, because of Stokes' theorem.

Consider $\mathcal{M}$ and a small deformation $\mathcal{M}'$:



$$\begin{aligned} \text{Then } U_\alpha(\mathcal{M}) U_\alpha^\dagger(\mathcal{M}') &= \exp\left(i\alpha \int_{\mathcal{M}} j - i\alpha \int_{\mathcal{M}'} j\right) \\ &= \exp\left(i\alpha \int_{\mathcal{M} - \mathcal{M}'} j\right) = \exp\left(i\alpha \int_{\partial B} j\right) = \exp\left(i\alpha \int_B dj\right) = 1 \end{aligned}$$

So

$$U_\alpha(\mathcal{M}) = U_\alpha^\dagger(\mathcal{M}')$$

(This should be understood at the level of path integrals, and as long as there are no charged operators in B .)

So in the abelian case we have, for example:

$$\begin{array}{ccc}
 \text{Diagram 1} & & \text{Diagram 2} \\
 \text{Circle with center } \bullet O(p) \text{ and boundary } S^1 & \xrightarrow[S^3 = \partial B_4]{=} & \text{Circle with center } \bullet O(p) \text{ and boundary } S^3 \\
 e^{i\alpha \int_{S^3} j} & & e^{i\alpha \int_{B^4} dj}
 \end{array}$$

$$\begin{aligned}
 &= O(p) + i\alpha \int_{B^4} O \, dj + \dots \\
 &= O(p) + \alpha \int_{B^4} S^4(p) \delta O + \dots \\
 &= O(p) + \alpha \delta O(p) + \dots \\
 &= O(p) + i\alpha q O(p) + \dots \\
 &= e^{i\alpha q} O(p)
 \end{aligned}$$