

2. TOPOLOGICAL PERFECT LINES IN 2D CFTs
~~via the d=1+1 symmetry category~~

Consider 2d CFTs that are

- Unitary
- Local $\otimes$ unique vacuum,
- Compact (discrete spectrum on my spatial slice)

$\exists T_{\mu\nu} / \partial^\mu T_{\mu\nu} = 0, T^\mu_\mu = 0.$

$\mathbb{R}^2$ Euclidean, $z, \bar{z}, (T_{zz}(z), \bar{T}_{\bar{z}\bar{z}}(\bar{z}))$
 $\partial T = 0 \quad \partial \bar{T} = 0$

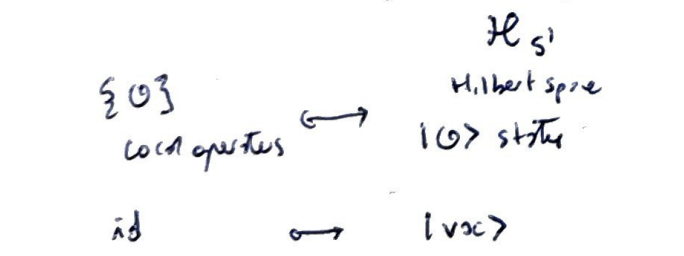
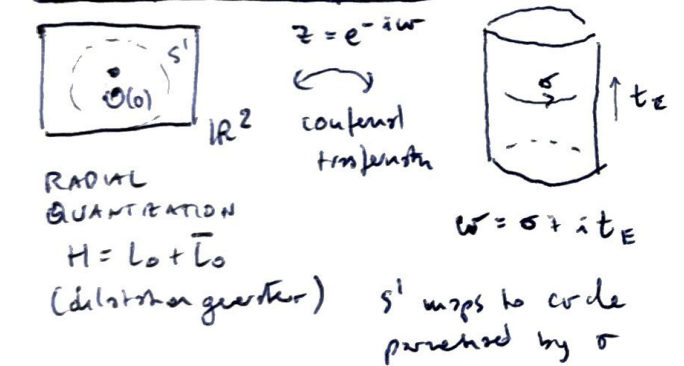
$$L_n = \oint \frac{dz}{2\pi i} T(z) z^{n+1}$$

$$\bar{L}_n = \oint \frac{d\bar{z}}{2\pi i} \bar{T}(\bar{z}) \bar{z}^{n+1}$$

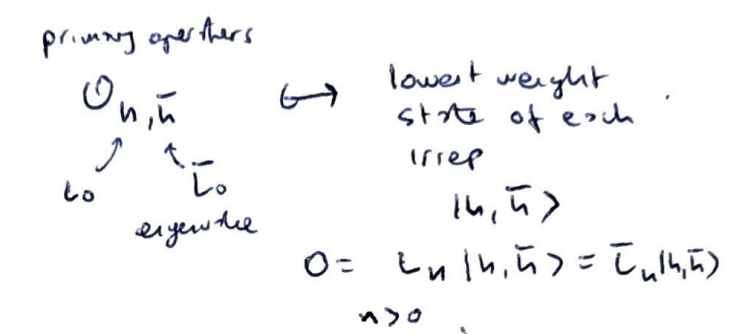
$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} \delta_{m,-n} n(n^2-1)$$

$\text{Vir}_L \times \text{Vir}_{\bar{L}}$

state-operator correspondence:



many constraints from unitarity property (character structure is inherited from Lorentz signature)



states in $\mathcal{H}_S^1$ have positive norms
 $\Rightarrow$ many unitarity constraints
 e.g. $\|L_{-2}|0>\| > 0 \Rightarrow c > 0$
 $\|L_{-1}|h,\bar{h}>\| > 0 \Rightarrow h, \bar{h} > 0$
 determine unitary irreps of $\text{Vir}_c (M_{c,h})$

For $c < 1$: discrete set of solutions (minimal models) Ising

$$c = 1 - \frac{6}{m(m+1)} \quad m = 3, 4, 5 \dots$$

discrete spectrum: Ising

$$h_{R,S}(m) = \frac{((m+1)\pi - m\pi)^2 - 1}{4m(m+1)}$$

$1 \leq r \leq m-1$
 $1 \leq s \leq r$ (w/ relation) $S \uparrow$

$c = \frac{1}{2}$ (Ising CFT) $m=3$

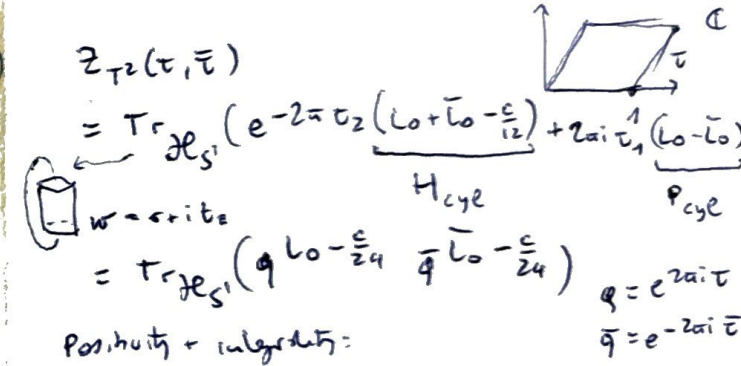
$h, \bar{h} = 0, \frac{1}{2}, \frac{1}{16}$

still need to constrain degeneracies and relation between h and $\bar{h}$

For that we use locality: in this context (Euclidean Heeg) locality means we can define the QFT on a general spacetime manifold and we are free to choose spatial slices to quantize the Heeg. Choices of these quantization will not affect the observables.

Considering the CFT on T^2 this leads to modular invariance as knots.

$$T^2 = \mathbb{C} / (2\pi i \mathbb{Z} \oplus \tau \mathbb{Z}) \quad \tau = \tau_1 + i\tau_2$$



Positivity + integrality:

$$Z_{T^2}(\tau, \bar{\tau}) = \sum_{h, \bar{h}} n_{h, \bar{h}} \chi_h(q) \bar{\chi}_{\bar{h}}(\bar{q})$$

modular invariance: $\tau \mapsto \tau + 1: T \rightarrow e^{2\pi i (h - \bar{h})} = 1$
 $\tau \mapsto -1/\tau: S$
 $\Rightarrow h - \bar{h} \in \mathbb{Z}$ (spin quantization)

Virasoro characters $= \text{tr}_{\mathcal{H}_{c,h}} q^{L_0 - \frac{c}{24}}$

For Ising it is a sum, but it can be much worse in general.

modular invariance $\Rightarrow S \cdot n \cdot S^T = n$

For Ising CFT: $c = \frac{1}{2}$ $n, \bar{n} = 0, \frac{1}{2}, \frac{1}{16}$

$n = \bar{n}$ by spin quantization so

$n, \bar{n}$ must be diagonal

unique vacuum: $n_{0,0} = 1$

there are only two further positive integers to fix

$c = \frac{1}{2}, S = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$ explicitly capital trans defect

$\Rightarrow n_{h,\bar{h}} = \delta_{h,\bar{h}}$ is the identity

$\Rightarrow \mathcal{H}_{S^1} = \left\{ 1, \frac{1}{2}, \frac{1}{16}, \frac{1}{16} \right\}$

PRIMARYS

Energy
operator

spin
operator

Ising CFT is completely solvable and one can calculate all the correlation functions. In particular, look at primaries and we see that

$\langle \sigma \sigma \sigma \rangle_{S^2} = \langle \sigma \varepsilon \varepsilon \rangle_{S^2} = \langle \dots \text{odd \# } \sigma \dots \rangle_{S^2} = 0$

these are Ward identities that clearly indicate the presence of a $\mathbb{Z}_2$ symmetry $\sigma \mapsto -\sigma$.

But there are more Ward identities that we can find:

$\langle \underbrace{\varepsilon \varepsilon \varepsilon \dots \varepsilon}_{\text{odd \#}} \rangle_{S^2} = 0$

but $\langle \sigma \sigma \varepsilon \rangle_{S^2} \neq 0$.

so this must be a vmlia $\mathbb{Z}_2$ symmetry.

Can we explain it using a topological operator?

$\mathbb{Z}_{\frac{1}{2}}^{\text{Ising}}(t, \bar{t}) = |\tau_0(t)|^2 + |\tau_{\frac{1}{2}}(t)|^2 + |\tau_{\frac{1}{16}}(t)|^2$

$\tau_0(t) = \frac{\sqrt{\Theta_3(t)} + \sqrt{\Theta_4(t)}}{2\sqrt{\eta(t)}}$

$\tau_{\frac{1}{16}}(t) = \frac{\sqrt{\Theta_2(t)}}{\sqrt{2}\sqrt{\eta(t)}}$

$\tau_{\frac{1}{2}}(t) = \frac{\sqrt{\Theta_3(t)} - \sqrt{\Theta_4(t)}}{2\sqrt{\eta(t)}}$

see Grouping 9108028

where

$\eta(t) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ $\Theta_3(t) = \sum_{n \in \mathbb{Z}} q^{n^2/2}$

$\Theta_2(t) = \sum_{n \in \mathbb{Z}} q^{(n-1/2)^2/2}$ $\Theta_4(t) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/2}$

to explain this we need to learn how to play with symmetry enriched operators in 2d CFT ~ internal image of free CFT.

reference: 1704.02330 Abhishhek Tachikawa
1802.04497 Chung-Li Shao
Wang, Yin

Philosophy: topological defect lines



$[D, T] = 0$

$\odot_D^0 = \delta \cdot \delta$

$\delta \cdot \delta$ has the same transformation properties of δ under vmlia.

$D_i \otimes D_j = \sum_k N_{ij}^k D_k$

Fusion ring



$\Rightarrow$ representation of fusion ring on $\mathcal{H}_{S^1}$

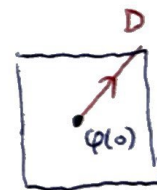
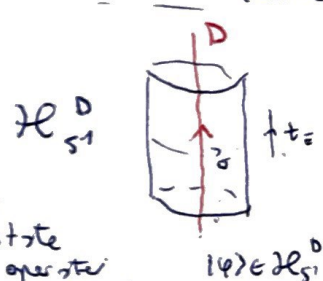
in particular

$\hat{D}|0\rangle = \langle \delta \rangle |0\rangle$

By locality:

we can extend topological line along time twisted Hilbert space

quantum dimension of D
 $\langle \delta \rangle = \sum_i \frac{1}{d_i} \text{unstable}$
 $\frac{1}{d_i} \text{unstable}$



D -twisted sector operator

(pointlike, but there is a line attached to it)

compatibility with fusion

$\Rightarrow N_{ij}^k \in \mathbb{Z}_{\geq 0}$

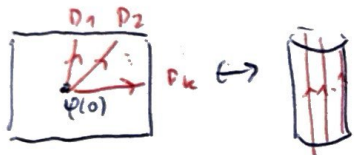
$\mathcal{H}_{S^1}^{D_i \otimes D_j} \cong \bigoplus_k \left(\underbrace{\mathcal{H}_{S^1}^{D_k} \otimes \dots \otimes \mathcal{H}_{S^1}^{D_k}}_{N_{ij}^k \text{ times}} \right)$

Moreover we want FAITHFUL action: $\exists \odot$ on which the topological line acts faithfully

$\odot \Rightarrow D \neq 1$ cannot end topologically

In general, we can have more defects coming together and isomorphisms

$$\mathcal{H}_{S^1}^{D_1, D_2, \dots, D_k} \simeq \mathcal{H}_{S^1}^{D_1 \oplus D_2 \oplus \dots \oplus D_k}$$



much of the information about the symmetry category is encoded in the isomorphism there

morphisms between topological defect lines are given by topological line operators



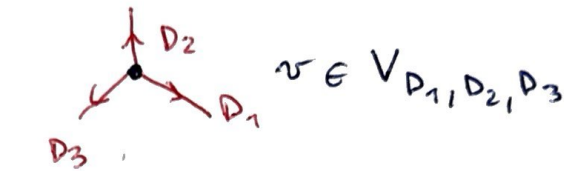
In our case we have that these belong to $\mathcal{H}_{S^1}^{D_1, \dots, D_k}$

these are the $h=\bar{h}=0$ subspace

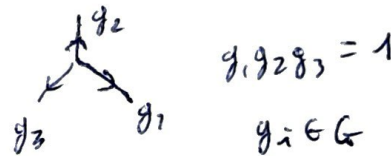
$$V_{D_1, \dots, D_k} \subset \mathcal{H}_{S^1}^{D_1, \dots, D_k}$$

Inherit grading by Virasoro.

many $\text{Hom}(D, 1)$ and $\text{Hom}(1, D)$ should be empty



is the structure that generalizes group multiplication



For group symmetries this is what happens.

there is always a dual defect:

$$\bar{D} \dagger = \dagger D$$

in general $\dim V_{D\bar{D}} \geq 1$, the defect line is simple if $\dim V_{D\bar{D}} = 1 \Rightarrow D \neq D_1 \oplus D_2$.

Fusion:

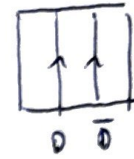
$$D_1 \oplus D_j = \sum_k \left(N_{ij}^k \right) D_k \quad \text{vs } D_j D_{j'} = D_{jj'}$$

In particular, $\dim V_{D_i D_j \bar{D}_k}$

$$D_2 \oplus \bar{D}_k = 1 \# + \dots$$

How to compute these?

consider T^2



$$= \# \square + \sum_k \square^{D_k}$$

$\downarrow \tau \rightarrow i\infty$

$$\dim V_{D\bar{D}} q^{-\frac{c}{24}} \bar{q}^{-\frac{\bar{c}}{24}}$$

$$(q^{L_0 - \frac{c}{24}}, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}) \quad \text{only } h=\bar{h}=0 \text{ survives}$$

some unit picks states

$$\# q^{-\frac{c}{24}} \bar{q}^{-\frac{\bar{c}}{24}}$$

unique vacuum

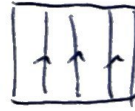
+ ... $D_k \uparrow$ $h, \bar{h} > 0$ suppressed $n, \bar{n} > 0$

$$\dim V_{D\bar{D}} = \#$$

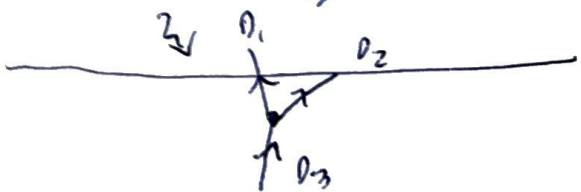
similarly:

$$\dim V_{D_i D_j \bar{D}_k} = N_{ij}^k$$

From consistency

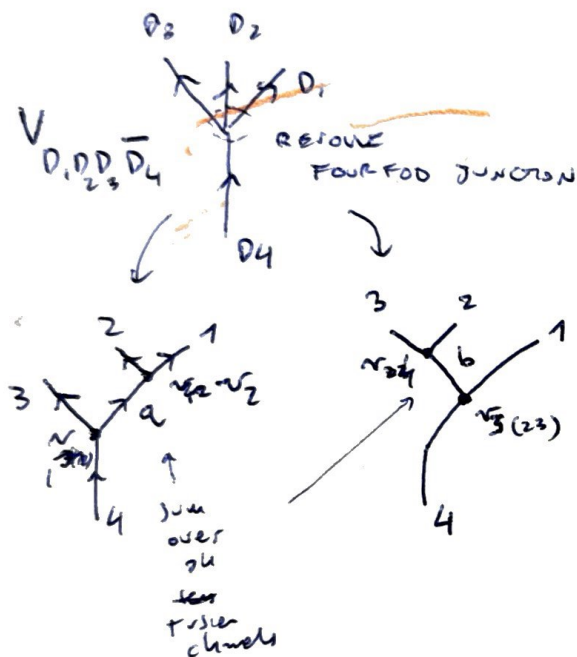


$v \in V_{D_1 D_2 \bar{D}_3} \leadsto$ morphism $D_1 \otimes D_2 \rightarrow \bar{D}_3$



important piece of data is given by F-symbols

there are already change of basis in junction vector spaces



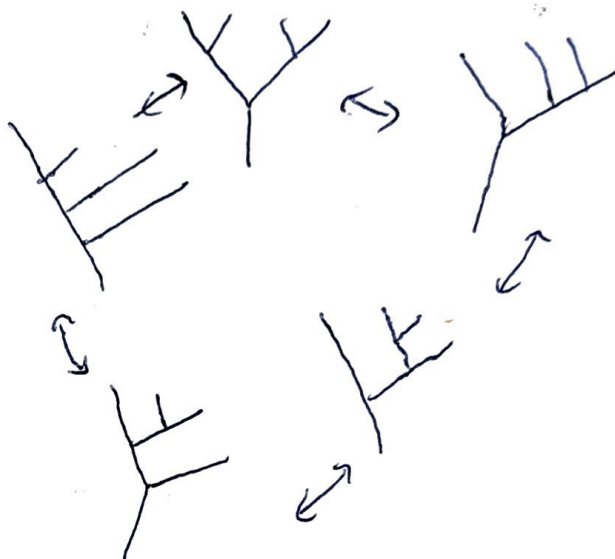
some Hilbert space $\rightarrow$ there is a unitary transformation between them

$$\begin{array}{c} 3 \\ \diagup \\ 2 \\ \diagdown \\ 1 \\ | \\ 4 \end{array} = \sum_b \begin{array}{c} 3 \\ \diagup \\ 2 \\ \diagdown \\ 1 \\ | \\ 4 \end{array} (F_4^{321})_{ab} (v_1 \otimes v_2, v_3 \otimes v_4)$$

F-move

they partition identity

$$V_{D_1 D_2 D_3 D_4 \bar{D}_5}$$



fusion condition

For usual symmetries / translatable cocycle condition on

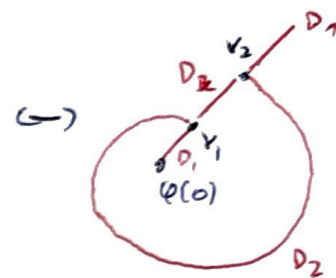
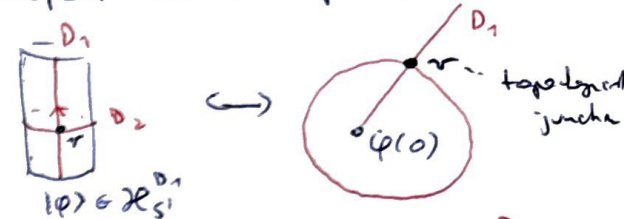
$$H^3(G, U(1))$$

characterize "nearly" when strictly capture nearly info.

forget just group law

Symmetries of one group and different members $\rightarrow$ wrapped in different fusion category.

action of symmetries on defect Hilbert space:



more generally:

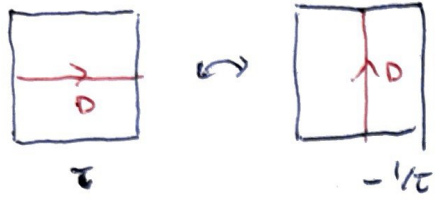
$$\mathcal{H}_{S^1}^{D_1} \rightarrow \mathcal{H}_{S^1}^{D_3}$$

IF symmetry is non-invertible there are more channels

maps between twisted Hilbert spaces $\sim$ (take algebra)

interesting to study and characterize explicitly, can be used to compute and determine explicitly the F-symbols.

interplay with locality: MODULAR BOOTSTRAP EQUATION 23



$$\text{tr}_{\mathcal{H}_S} (D q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}})$$

$$= \text{tr}_{\mathcal{H}_S^0} (\tilde{q}^{L_0 - \frac{c}{24}} \bar{\tilde{q}}^{\bar{L}_0 - \frac{c}{24}})$$

$$\tilde{q} = e^{2\pi i (-\frac{1}{\tau})} = \sum_{i,j} n_{i,j}^0 \chi_i(\tilde{q}) \bar{\chi}_j(\tilde{q}) \in \mathbb{Z}_{\geq 0}$$

LHS: action of D on the Hilbert space

RHS: structure of Virasoro representations $\rightarrow$ descriptions in Virasoro blocks

example Ising model:

LHS: $A_{id} |\tau_0|^2 + A_\epsilon |\tau_{1/2}|^2 + A_\sigma |\tau_{1/16}|^2$

↑
eigenvalues of
action of operator
on primaries

RHS: same characters with different modular parameter, related by symmetry transformation

you can read off the solution from modular data:

there are only 3 independent columns to this modular transformation:

	A_{id}	A_ϵ	A_σ
id	1	1	1
η	1	1	-1
N	$\sqrt{2}$	$-\sqrt{2}$	0

can read off fusion algebra & twisted Hilb. spaces

HAS A KERNEL (HALLMARK ON NON-INV.)

$$N^2 = 11 + \eta$$

$$\eta^2 = 11$$

$$N\eta = \eta N = N$$

$$\langle N \rangle = \sqrt{2}$$

$$\text{Diagram with } N \text{ and } \sqrt{2} \Rightarrow \langle SSS \rangle = 0$$

$$\text{Diagram with } N \text{ and } \sqrt{2}$$

Moreover:

$$\text{Diagram with } N \text{ and } \sqrt{2} = \sqrt{2} \cdot \text{Diagram with } \mu$$

$$\mathcal{H}_{S^1}^\eta = \{ \psi_{\frac{1}{2},0}, \tilde{\psi}_{0,\frac{1}{2}}, \mu_{\frac{1}{16},\frac{1}{16}} \}$$

$$\mathcal{H}_{S^1}^N = \{ s_{\frac{1}{16},0}, \tilde{s}_{0,\frac{1}{16}}, \lambda_{\frac{1}{16},\frac{1}{2}}, \hat{\lambda}_{\frac{1}{2},\frac{1}{16}} \}$$

only one vertex is relevant
just as $V_{NN\eta}, V_{N\eta\eta}, V_{\eta\eta\eta}$

Using F-symbols:

$$\text{Diagram with } N \text{ and } N = \frac{1}{\sqrt{2}} \left(\text{Diagram with } \eta \text{ and } \eta + \text{Diagram with } \eta \text{ and } \eta \right)$$

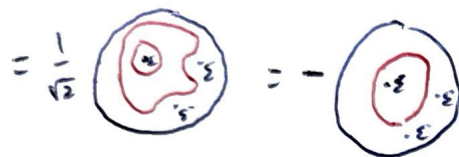
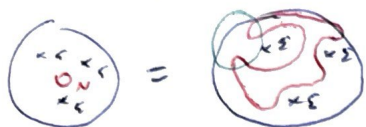
$$\text{Diagram with } N \text{ and } N = \frac{1}{\sqrt{2}} \left(\text{Diagram with } \eta \text{ and } \eta - \text{Diagram with } \eta \text{ and } \eta \right)$$

$$\text{Diagram with } N \text{ and } N = - \text{Diagram with } N \text{ and } N$$

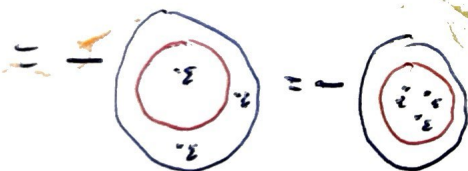
map between $\mathcal{H}_{S^1}$ and $\mathcal{H}_{S^1}^\eta$

possible because in the twisted Hilb. space we see $\sum w_i \chi_i$

bubble out N :



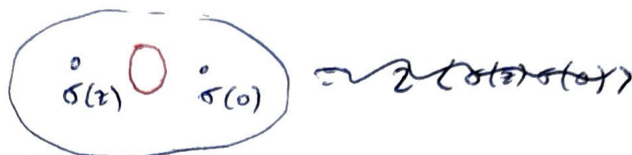
because in the de^2 twisted sector there is no operator with conformal dimension $1/2, 1/2$.



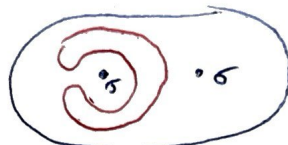
$\Rightarrow$ odd # of Σ gives zero.

bubble N away

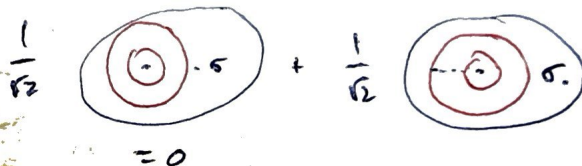
Consider this:



"



"



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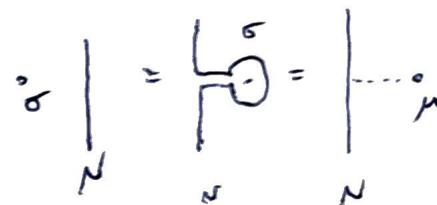
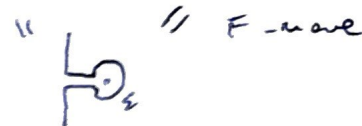
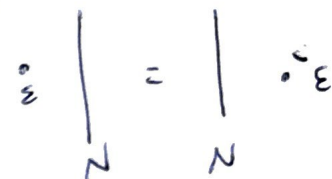


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Fix coefficient to recover order/dimension formula.

topological defect passing through operators.



$N \Rightarrow$ CFT is self dual under $\mathbb{Z}_2$ -orbifold.

what is a $\mathbb{Z}_2$ -orbifold?

orbifold $\equiv$ g-gauge.

How does one gauge a finite group symmetry?

In general when you can gauge a symmetry you couple it to backgrounds and you sum over all possible choices.

(2-4)

It is the one in a
dozen of your world
with a consistent collection
of rejected operators.



no background



specific choice of background.

The p-value function of $\log/\mathbb{Z}_2$ on $\mathcal{X}$ is obtained by this operation:

$$z_{T2}^{1\text{sing}/2} = z_{T2}^{1\text{sing}} + z_{T2}^{1\text{sing}}$$

$$+ z^{159} + z^{159}$$

$$= \left(\begin{array}{cc} \boxed{} + \boxed{} & \boxed{} \\ \boxed{} + \boxed{} & \boxed{} \end{array} \right) \frac{1}{2}$$

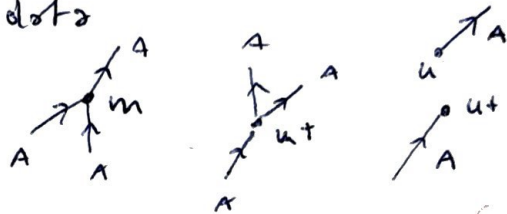
normalise to have a single value

modular
insurance
requires
all the
four
grounds

notice that this is the same as describing the terms with a non-simple

line: $A = \frac{1}{2}(1 + \eta) R^{1/2}$

together with some extra
dots



(in this case this detects the normality of the problem factor).

Now consider Ising on T^2

$$\begin{aligned} Z_{T^2}^{\text{sing}} &= \square = \square_{\text{ON}} / \langle N \rangle \\ &= \square_{\text{ON}} / \langle N \rangle \\ &= \square_{\text{ON}} / \langle N \rangle \\ &= \square_{\text{ON}} / \langle N \rangle \\ &= \square_{\text{ON}} / \langle N \rangle \\ &= Z_{T^2}^{\text{sing}} / \langle N \rangle \end{aligned}$$

The defect is -

The logical operator

$$N^2 = 1 + \eta$$

expresses the self directly

Rule:

This gives us $TY(G)$

fusion gehen:

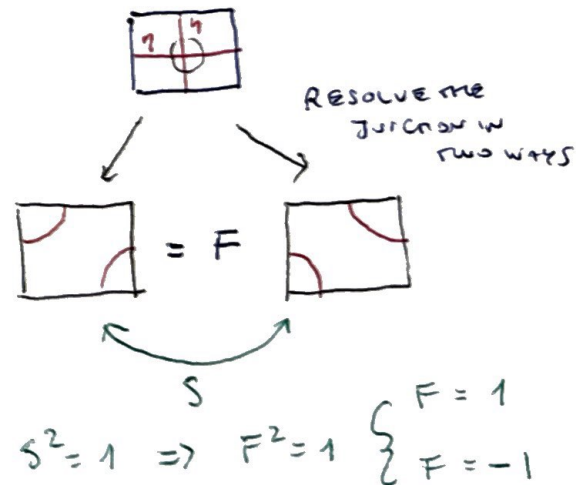
$$N^2 = \sum_{g \in G} L_g$$

$$L_g L_{g'} = L_{gg'}$$

$$NL_g = L_g N = N$$

2d Heavy with this symmetry
is invariant upon gauging
the discrete symmetry G .

convert or modify ($\mathbb{Z}_2$ case)



when $F=+1$, the $\mathbb{Z}_2$ symmetry is gaugable because you can form a modular invariant orbifold partition function.

when $F=-1$, there is a clash with modularity because

$$\begin{array}{|c|c|} \hline \hline \hline \end{array} = - \begin{array}{|c|c|} \hline \hline \hline \end{array}$$

the only way to see the deg would be if

$$\begin{array}{|c|} \hline \hline \hline \end{array} = 0 \text{ but then}$$

$$\xrightarrow{T} \begin{array}{|c|} \hline \hline \hline \end{array} = 0$$

$$\Rightarrow \begin{array}{|c|} \hline \hline \hline \end{array} = 0$$

but we know
is wrong
because
of orth.

For ordinary symmetries the anomaly is captured by the F-symbol $\rightarrow$ smooth.

For non-invertible symmetry it is more subtle.

Even if F is unbroken, one can still find a relevant object $(A, u, u^\dagger, u, u^\dagger)$

That obeys

$$\begin{array}{c} | \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ | \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \\ | \end{array}$$

(Frobenius property).

$\rightarrow$ then A can be gapped.

Final comment on 2d: dynamical consequence of non-invertible sym

$$T_{UV} + \int d^2x \mathcal{O}_{u, \bar{u}} = h$$

at D-term $\Delta \leq 2$

$\rightarrow T_{IR}$ relevant deformation

$$0 \cdot 1_0 = 1_0 \cdot 0$$

$$\text{nd } \langle D \rangle \notin \mathbb{Z}_+$$

then T_{IR} cannot be trivially gapped. ($\Rightarrow$ either SSB or CFT)

Proof: by contradiction: since T_{IR} is gapped trivially, i.e. it has a unique ground state

Consider

$$\begin{array}{|c|} \hline \hline \hline \end{array} = \begin{array}{|c|} \hline \hline \hline \end{array}$$

$$\langle D \rangle$$

$$\begin{array}{|c|} \hline \hline \hline \end{array}$$

$$\text{tr } \rho_{S^1}^D(1)$$

$$\Rightarrow \langle D \rangle \in \mathbb{Z}_+$$

contradiction.

trivially gapped theory has

example: Prochiral log, $C = \frac{1}{2} \log$, $\int [\frac{1}{2} \frac{1}{x}, N] = 0 \rightarrow R \rightarrow \text{gapped (log)}$
 $\rightarrow \text{gapped (SSB)}$

RMK: notice that $\langle D \rangle$ can be

computed from F symbols, so it cannot reverse
(discrete set of solutions, rigid structure that does not change deg RG).

$$\text{in Ising, } \log \pm \int d^2x \frac{1}{2} \frac{1}{x}$$

+ trivial
- we know $\mathbb{Z}_2$ SSB

$\rightarrow$ denumerable between the two.

Symmetry

$$T_4(\mathbb{Z}_2)$$

$$W^2 = 1 + W$$

$$\langle W \rangle = \frac{1 + F}{2}$$