The fuzzball paradigm

Lecture 1

Samir D. Mathur

The Ohio State University

Plan:

I: The puzzle:

- A first pass
- The small corrections theorem

II: Resolving the puzzle:

- The fuzzball paradigm

III: The causality constraint:

- Breakdown of the semiclassical approximation
- A picture of gravitational collapse
- The flaw in the firewall argument
- Cosmology: the big bang

Gravity is an attractive force

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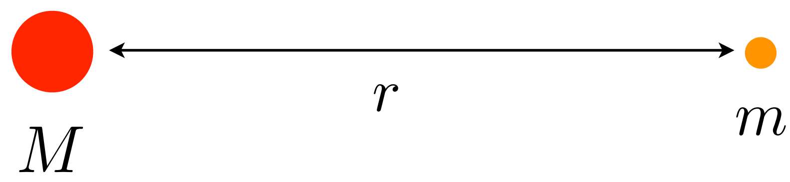


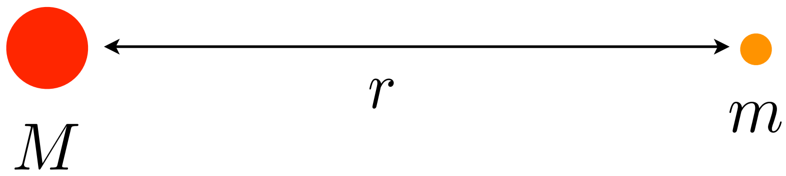
$$PE = -\frac{GMm}{r}$$

By itself, the small mass has an intrinsic energy

$$E = mc^2$$

When it is placed near the larger mass, what energy should we assign ?
Let us start with the Newtonian approximation ...

$$E = mc^2 - \frac{GMm}{r}$$

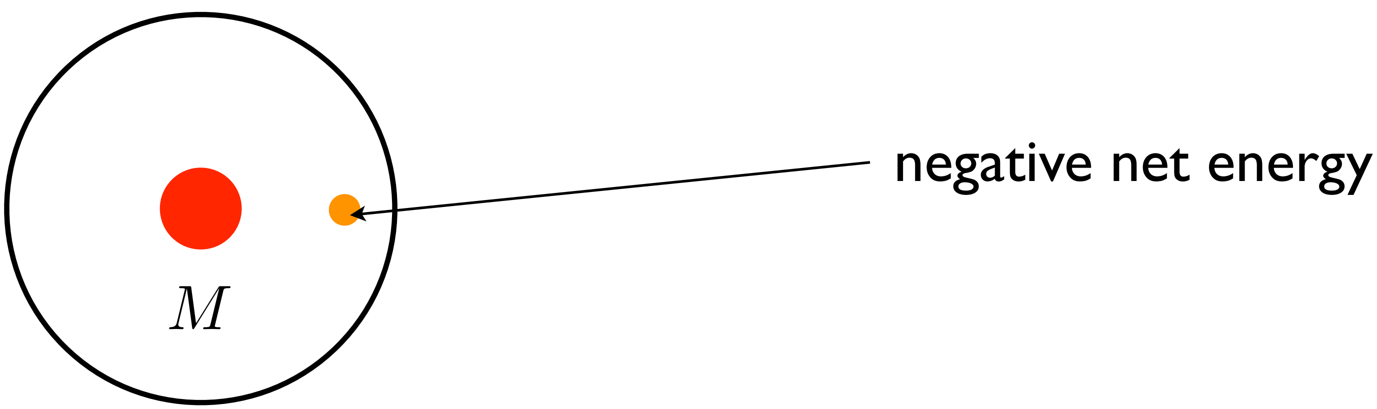


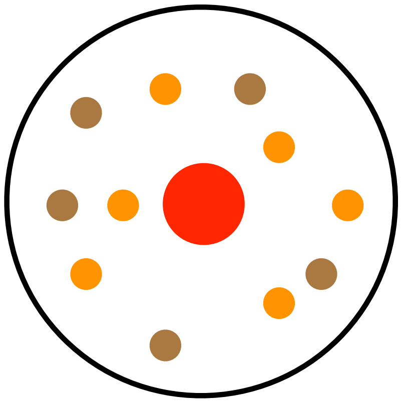
$$E = mc^2 - \frac{GMm}{r}$$

We see that the total energy of m becomes zero at

$$r = \frac{GM}{c^2}$$

and for smaller r it is negative





Suppose we keep placing more and more masses inside the horizon radius, until the mass comes to zero (or close to zero; it may stop at the planck mass)

Then we have a low mass object with a lot of internal structure

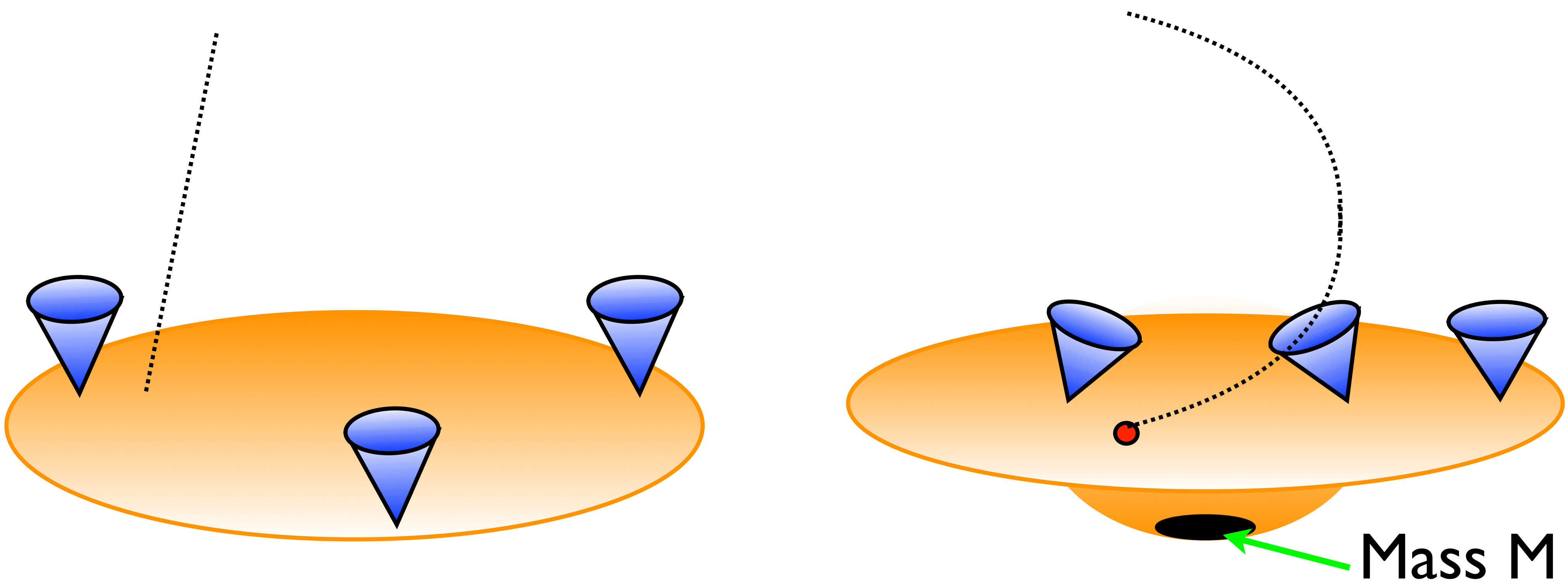
The problem is that we can make an infinite number of such planck mass objects, because we can start with an arbitrarily large mass…

Of course since we have talked about Gravity and Special Relativity, we should really use General Relativity

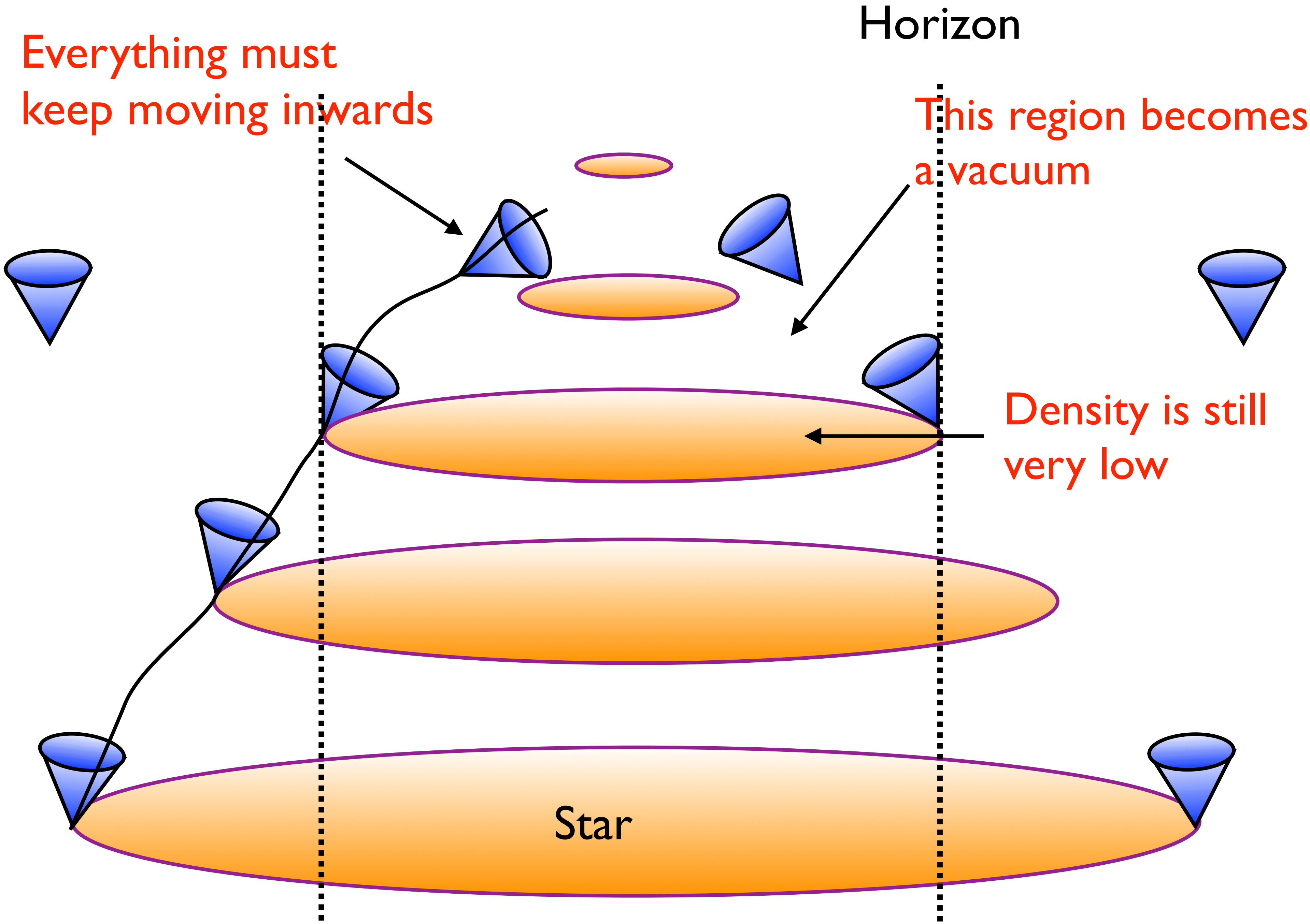
General relativity

Mass curves spacetime

All the ‘force’ of gravity is encoded in this curvature of spacetime



The black hole



Why is it hard to stop black hole formation?

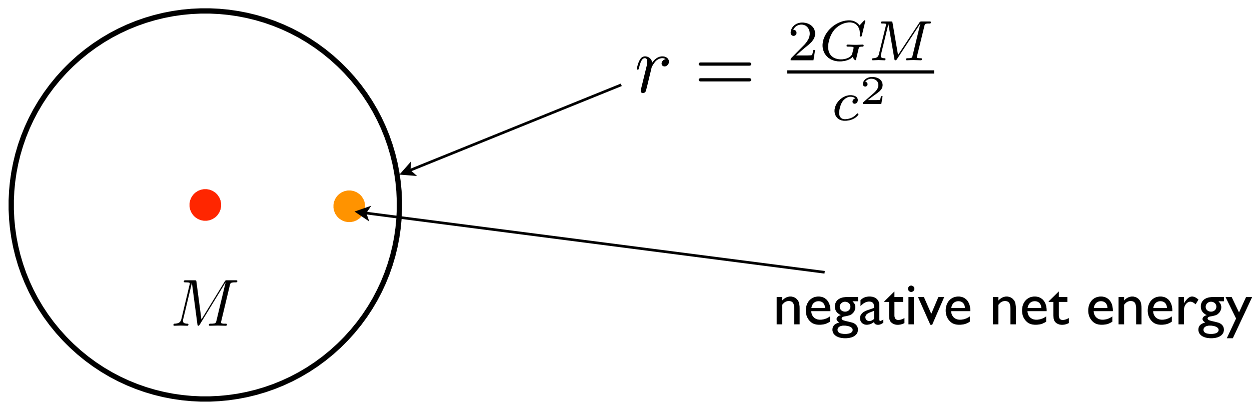
Horizon radius $$R = \frac{2GM}{c^2}$$

Flat space approximation $$M = \frac{4\pi}{3}\rho R^3$$

$$R = \frac{2G}{c^2}\left(\frac{4\pi}{3}\rho R^3\right) \Rightarrow \rho = \left(\frac{3c^2}{8\pi G}\right)\frac{1}{R^2}$$

We see that as we take bigger and bigger holes, the density at the point of horizon formation becomes smaller and smaller

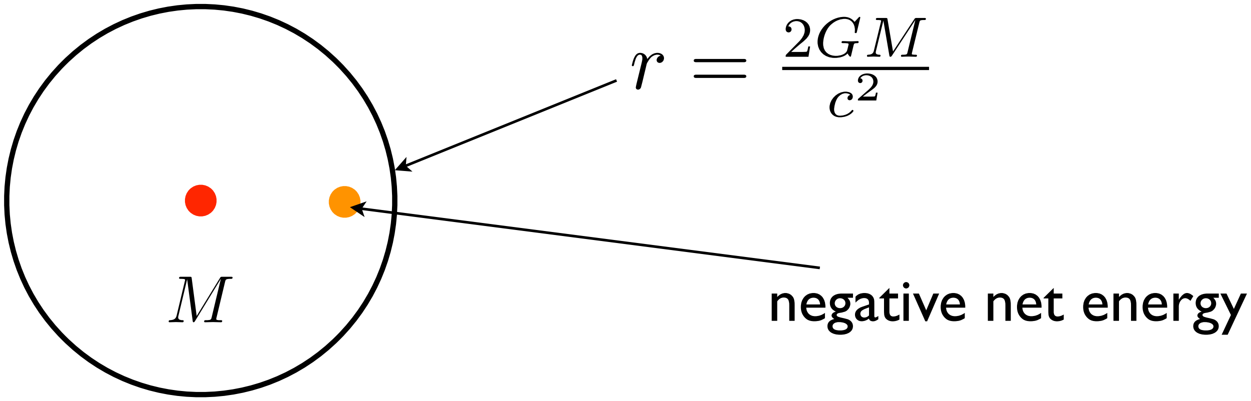
Repeating our argument properly with general relativity does not change our answer much



Q: Do we have a region where energy is negative?

Q: If not, then how do we stop horizon formation?

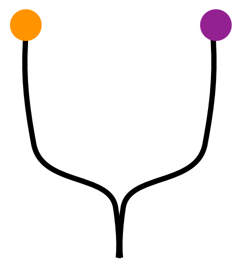
Hawking radiation

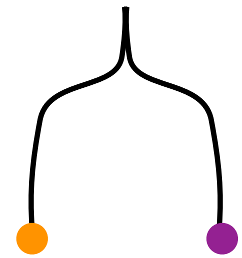


But how should we place the particle inside the hole ?

Hawking: Quantum mechanics will do this automatically …

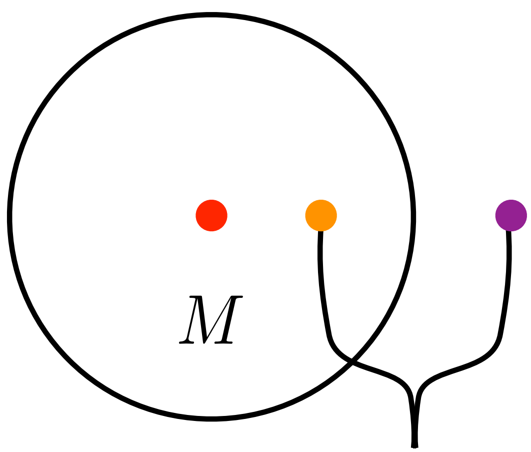
In quantum mechanics, the vacuum can have fluctuations which produce a particle-antiparticle pair





$$\Delta E \, \Delta t \sim \hbar$$

But if a fluctuation happens near the horizon, the particles do not have to re-annihilate



$$\Delta E = 0 \quad \rightarrow \quad \Delta t = \infty$$

Thus the negative energy particle gets automatically placed in the correct position inside the horizon

The outer particle drifts off to infinity as 'Hawking radiation'

The mass of the hole has gone down, so the horizon shrinks slightly

The process repeats, and another particle pair is produced

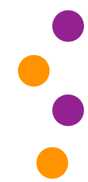
The energy of the hole reduces

But overall energy of hole + radiation is conserved

Two possibilities:

(a) Black hole evaporates away completely

vacuum



(b) A planck sized remnant is left

We will see that each of these possibilities presents us with problems …

Q: If we have a horizon, then can we stop or alter the process of Hawking evaporation? If so, how?

Q: If the evaporation is not stopped or altered, then

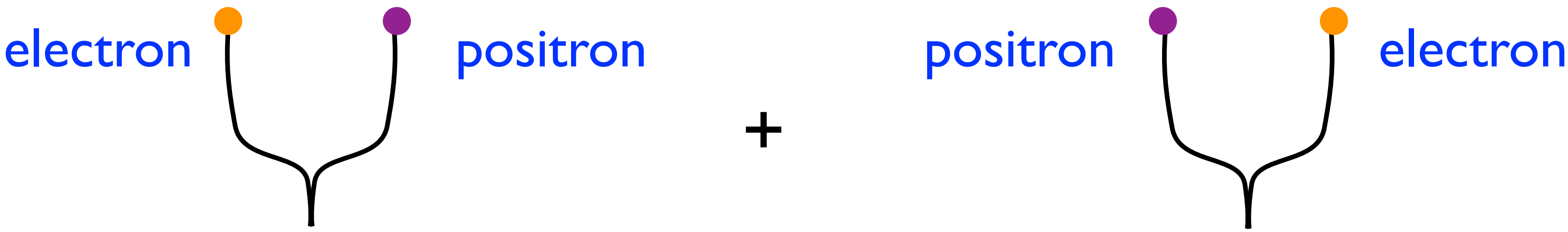
(a) does the hole evaporate away completely?

or

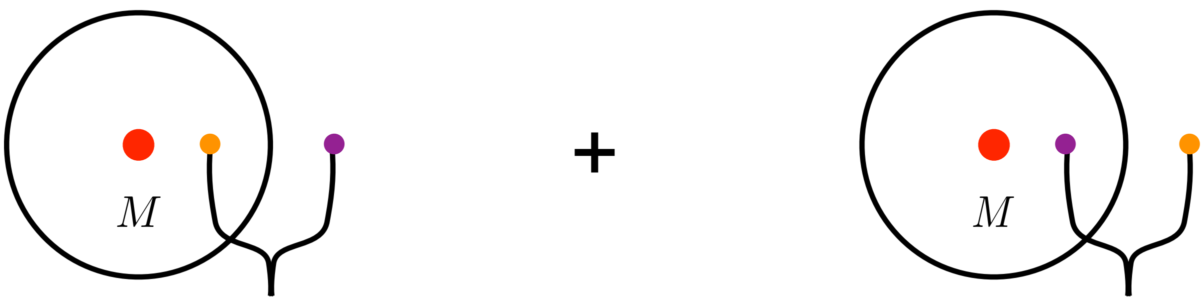
(b) Are we left with a planck sized remnant?

The problem with Hawking evaporation

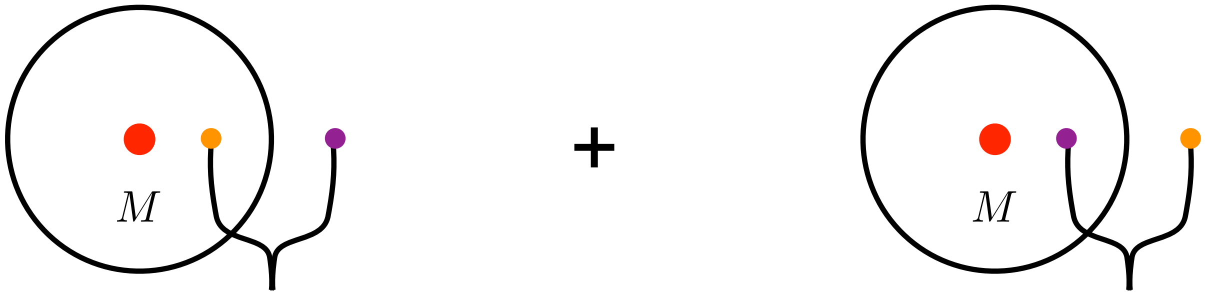
The crucial issue now has to do with ‘entanglement’



Vacuum fluctuations typically produce entangled states ...



So the state of the radiation is entangled with the state of the remnant



The entanglement can come from many sources:

(a) Charge: positive and negative $\quad e^+e^- + e^-e^+$

(b) Spin: Up and down $\quad \uparrow\downarrow - \downarrow\uparrow$

(c) Existence or nonexistence of a particle

$$C_0|0\rangle|0\rangle + C_1|1\rangle|1\rangle + C_2|2\rangle|2\rangle + \dots$$

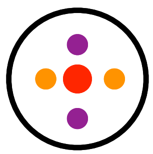
General form of entangled state

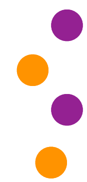
$$|\Psi\rangle = \sum_{i=1}^{M} C_i \; \psi_i^A \otimes \chi_i^B$$

The amount of this entanglement is very large ...

If N particles are emitted, then there are 2^N possible arrangements

We can call an electron a 0 and a positron a 1





	111111	000000
	....	....
+	010011	101100
+	000000	111111

Now there are two possibilities:

(a) Information loss: The evaporation goes on till the remnant has zero mass. At this point the remnant simply vanishes

vacuum

000000

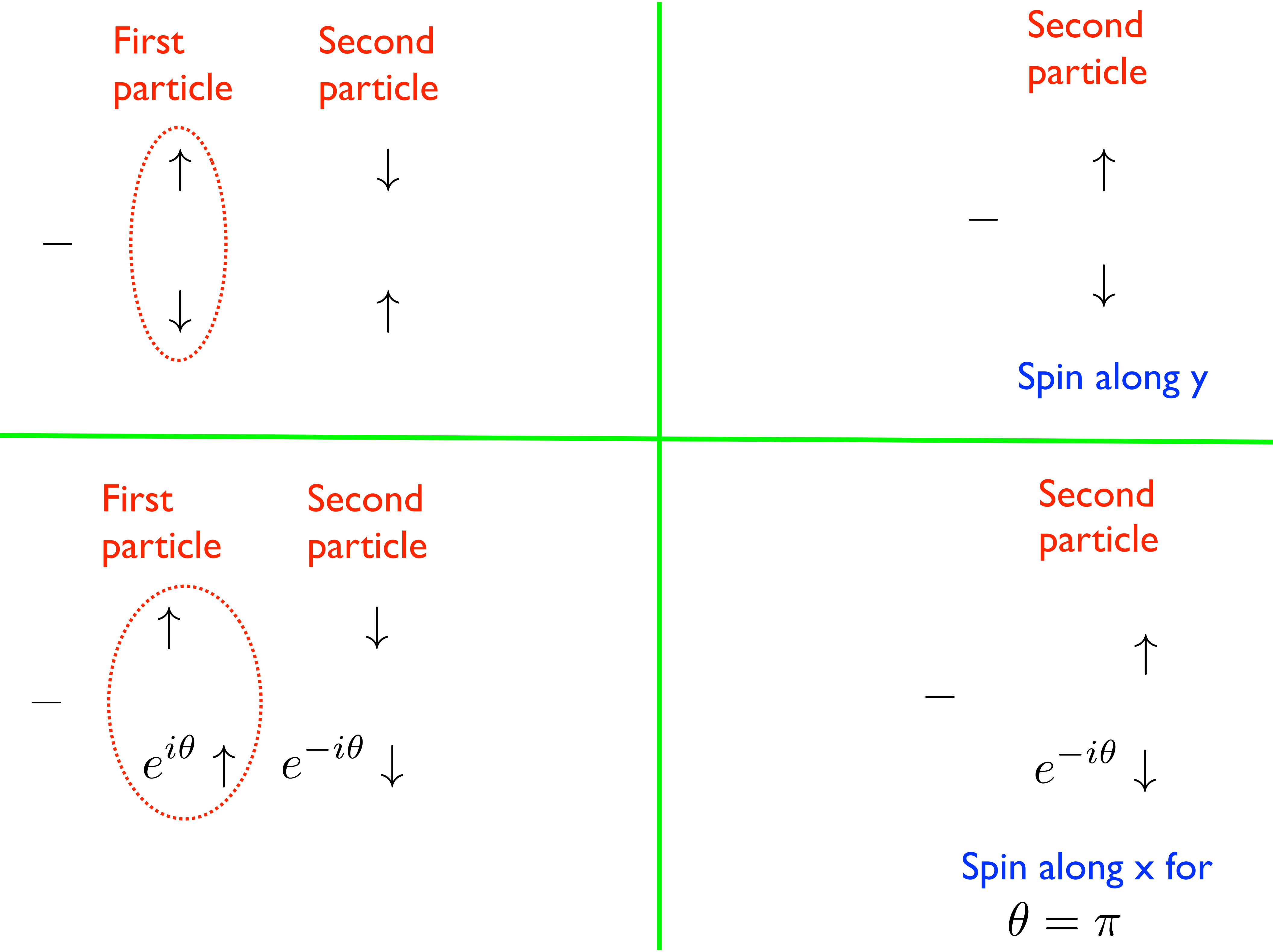
....

101100

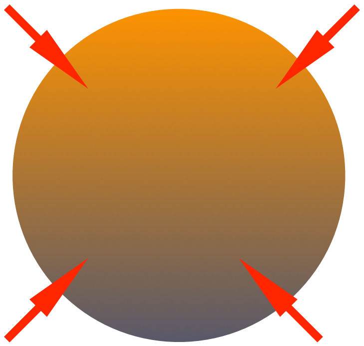
111111

The radiation is entangled, but there is nothing that it is entangled WITH

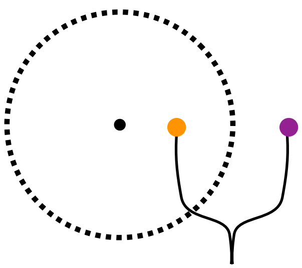
The radiation cannot be assigned ANY quantum state ... it can only be described by a density matrix ... this is a violation of quantum mechanics (Hawking 1975)


First particle
Second particle
$\uparrow$ $\downarrow$
$-$
$\downarrow$ $\uparrow$
Second particle
$\uparrow$
$-$
$\downarrow$
Spin along y
First particle
Second particle
$\uparrow$ $\downarrow$
$-$
$e^{i\theta}\uparrow$ $e^{-i\theta}\downarrow$
Second particle
$\uparrow$
$-$
$e^{-i\theta}\downarrow$
Spin along x for
$\theta = \pi$

The black hole information paradox



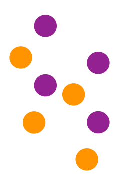
Star is described by
a wavefunction Ψ_i



$$\frac{1}{\sqrt{2}}(00+11)$$

Quantum mechanics:
entangled pairs are created

vacuum

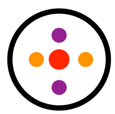


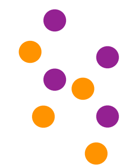
Cannot be described by
any quantum state

Contradicts quantum theory $\quad \Psi_f = e^{-iHt}\Psi_i$

(b) We assume the evaporation stops when we get to a planck sized remnant.

The remnant must have at least 2^N internal states





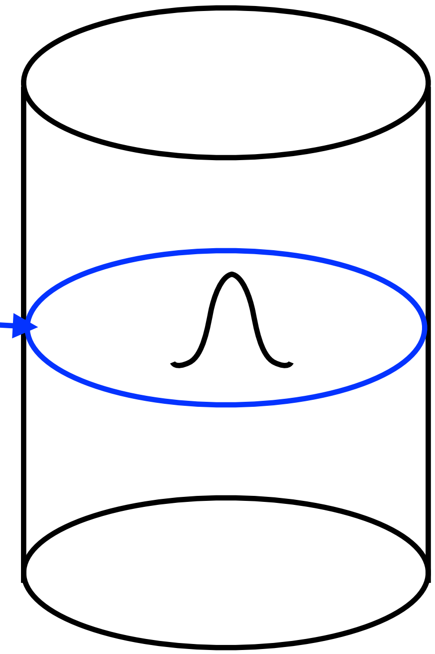
	111111	000000
	....	....
+	010011	101100
+	000000	111111

Can we hold an unbounded number of states in planck volume with energy limited by planck mass?

But if we accept AdS/CFT duality then we cannot have remnants

CFT with finitely many degrees of freedom in a finite volume

So CFT has finitely many states with energy less than E



AdS/CFT duality (Maldacena 97)

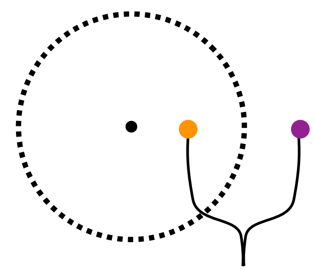
The CFT lives on a sphere of radius R_{CFT}

The AdS has a curvature radius R_{AdS}

An excitation of energy $1/R_{AdS}$ in the center of AdS has an energy $1/R_{CFT}$ in the CFT

So a planck mass remnant in AdS corresponds to finitely many states in the CFT; thus we cannot have remnants

Q: Does entanglement keep rising between the emitted radiation and the remaining hole? If not why not?

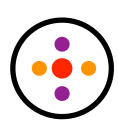


$$\frac{1}{\sqrt{2}}(00+11)$$

Q: Does the hole completely evaporate away? If so, is there any entanglement left at the point of evaporation?

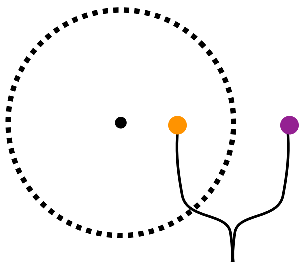
vacuum

Q: Is a remnant left? If so, do we accept that there is no analogue of AdS/CFT duality?



Can we prevent the formation of a horizon?

Given how much trouble we are having because of a horizon, let us ask if we can prevent the formation of a horizon



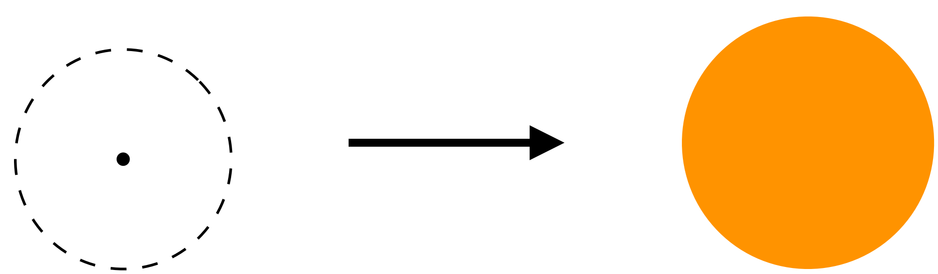
$$\frac{1}{\sqrt{2}}(00+11)$$

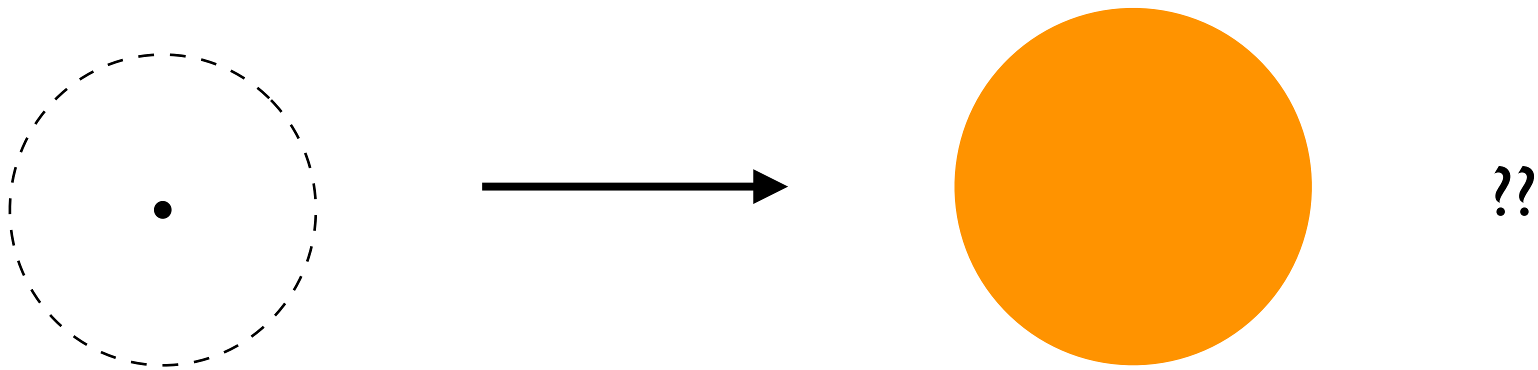
We have already seen that for a large star, the density is very low at the point of horizon formation

$$\rho = \left(\frac{3c^2}{8\pi G}\right)\frac{1}{R^2}$$

so it seems hard to stop the collapse.

But can we quantify this difficulty?





This is not easy to get: matter near the horizon is pulled in very strongly, and falls in

Buchdahl's theorem: If we have a perfect fluid ball with radius

$$R < \frac{9}{4} M$$

then the pressure will have to diverge somewhere

Metric

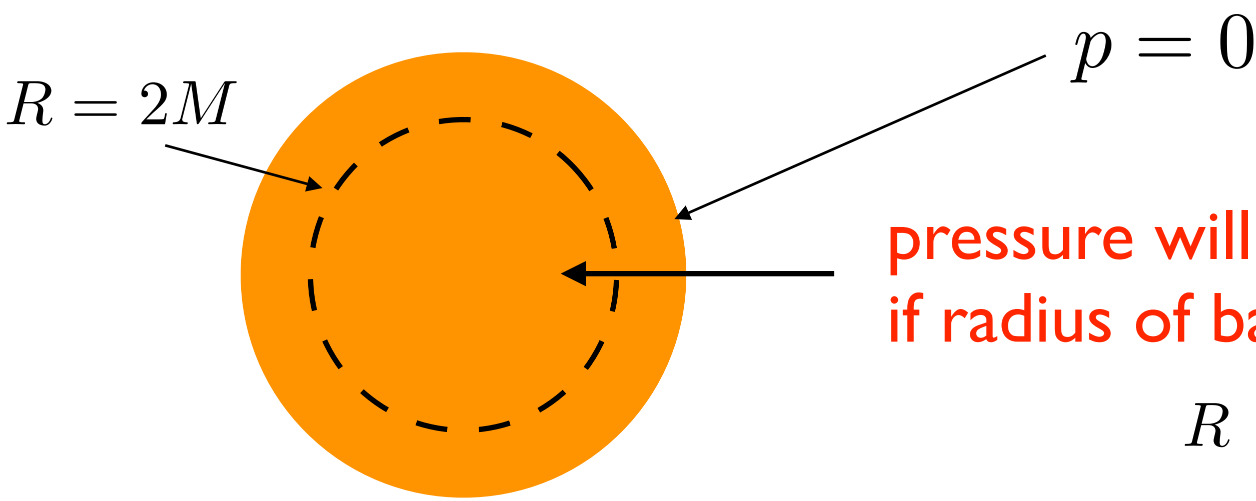
$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Stress tensor

$$T_{\hat{\mu}\hat{\nu}} = diag\{\rho(r), p(r), p(r), p(r)\}$$

Basic equation: The gradient of pressure has to balance the attraction of gravity

$$p'(r) + \frac{\alpha'(r)}{2}(\rho(r) + p(r)) = 0$$

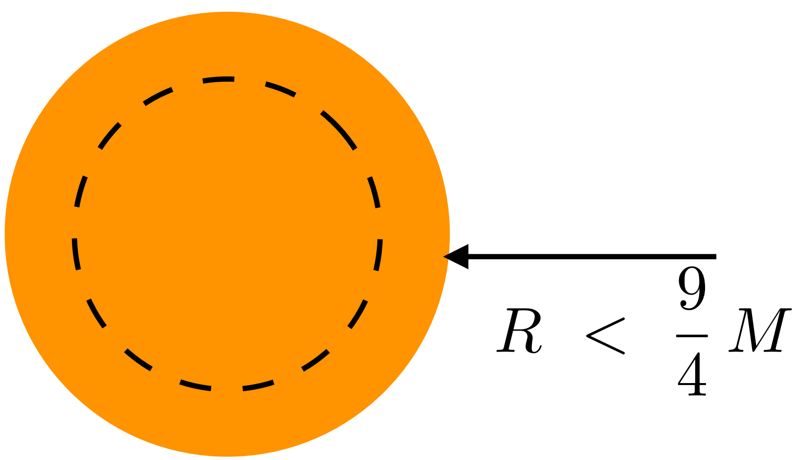


pressure will diverge somewhere if radius of ball is

$$R < \frac{9}{4}M$$

We can argue that if $p \to \infty$ somewhere, then the solution is unphysical

Thus if the ball reaches a radius $R < \frac{9}{4}M$ then it must continue to collapse through $R = 2M$ and create a horizon



Q: If stars are allowed to reach a size $R < \frac{9}{4}M$ then can collapse be prevented? If so, how?

Can we deform the horizon ?

If we do form a horizon, then can we deform it (so that the pair creation process may change) ?

The no-hair theorem (Bekenstein):

Try to add a time-independent classical scalar field to the hole

Time independent metric $$ds^2 = -f(y)dt^2 + h_{ij}(y)dy^i dy^j$$

Lagrangian for scalar field $$L = -\frac{1}{2}\partial_a\phi\partial^a\phi - V(\phi)$$

Assume potential is stable $$V'' < 0$$

Then the only solution is $$\phi = \phi_0 = constant$$

So the geometry of the hole does not deform (Classical no-hair theorem)

Can we change the quantum state of the hole, without changing the classical geometry ?

Hawking radiation is a quantum process, and if we change the quantum state around the horizon, then maybe the pair creation process will be altered

Consider a free scalar field in flat spacetime

$$\Box\phi = 0$$

The solutions to this equation are

$$f = e^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \text{with} \qquad \omega = |\vec{k}|$$

To quantize this scalar field, we write

$$\hat{\phi} = \sum_{\vec{k}} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega}} \left(\hat{a}_{\vec{k}}\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} + \hat{a}^{\dagger}_{\vec{k}}\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} \right)$$

and impose the commutation relations

$$[\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'}$$

Then each fourier mode becomes a harmonic oscillator with frequency

$$\omega = |\vec{k}|$$

The state annihilated by all the annihilation operators is the vacuum $|0\rangle$

$$\hat{a}_{\vec{k}}|0\rangle = 0$$

Acting with a creation operator $\hat{a}^\dagger_{\vec{k}}$ adds a particle of momentum $\vec{k}$ and energy $\omega = |\vec{k}|$

We can add many particles: $\hat{a}^\dagger_{\vec{k}_1} \ldots \hat{a}^\dagger_{\vec{k}_n} |0\rangle$

We can make a coherent state $e^{\mu_{\vec{k}} \hat{a}^\dagger_{\vec{k}}} |0\rangle$

where $\mu_{\vec{k}}$ is a complex number

For large $|\mu_{\vec{k}}|$, the expectation value $\langle \hat{\phi} \rangle$ behaves like a classical field solution to the wave equation

$$\Box \phi = 0$$

(This the relation between the quantum and the classical theory)

The same formalism works in curved spacetime. The wave-equation is

$$\Box\phi = g^{ab}\phi_{;ab} = 0$$

We take a complete set of solutions $f_i(x)$ to the wave equation

$$\Box f_i = 0$$

and we write $$\hat{\phi} = \sum_i \left(\hat{a}_i f_i(x) + \hat{a}_i^\dagger f_i^*(x) \right)$$

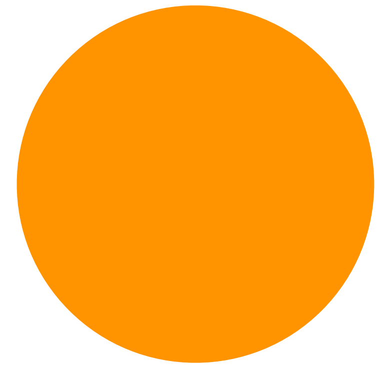
We again impose the commutation relations $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}$

The vacuum $|0\rangle$ is given by $\hat{a}_i|0\rangle = 0$

Particle excitations are given by $\hat{a}_{i_1}^\dagger \ldots \hat{a}_{i_n}^\dagger |0\rangle$

Adding 'hair' to a star:

We solve the wave equation in this background

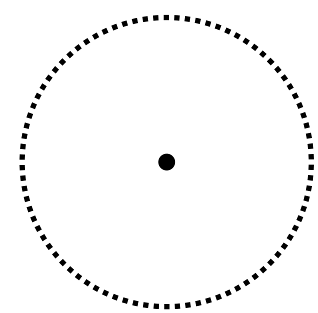


$$\Box\phi = 0 \ , \qquad \phi = Y_{lm} f_{lmn}(r) e^{-i\omega t}$$

We then write the quantum field as

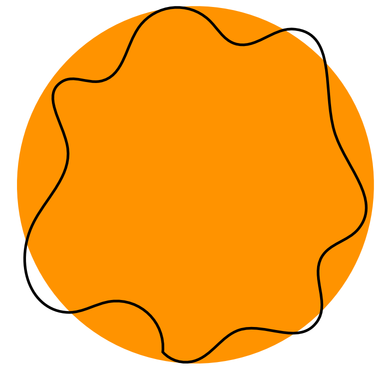
$$\hat{\phi} = \sum_{l,m,n} \left(\hat{a}_{lmn} Y_{lm}(\theta, \phi) f_{lmn}(r) e^{-i\omega t} + \hat{a}^{\dagger}_{lmn} Y^{*}_{lm}(\theta, \phi) f^{*}_{lmn}(r) e^{i\omega t} \right)$$

The vacuum state $|0\rangle$ of the quantum field is given by



$$\hat{a}_{l_i m_i n_i} |0\rangle = 0$$

We can change the state by adding quanta



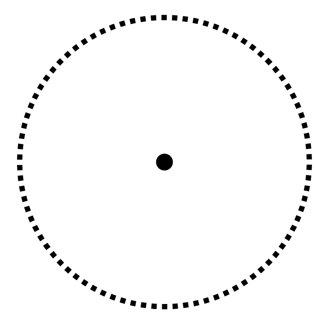
$$\hat{a}^\dagger_{l_1 m_1 n_1} \dots \hat{a}^\dagger_{l_1 m_1 n_1} |0\rangle$$

This adds 'hair' to the star. If we take a coherent state

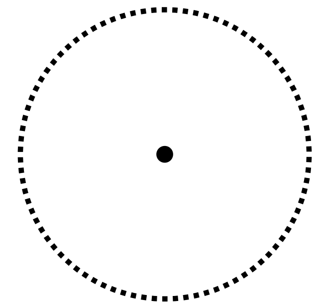
$$e^{\mu_{lmn} \hat{a}^\dagger_{lmn}} |0\rangle$$

then we get 'classical' deformations of the star.

But all his does not work if there is a horizon …



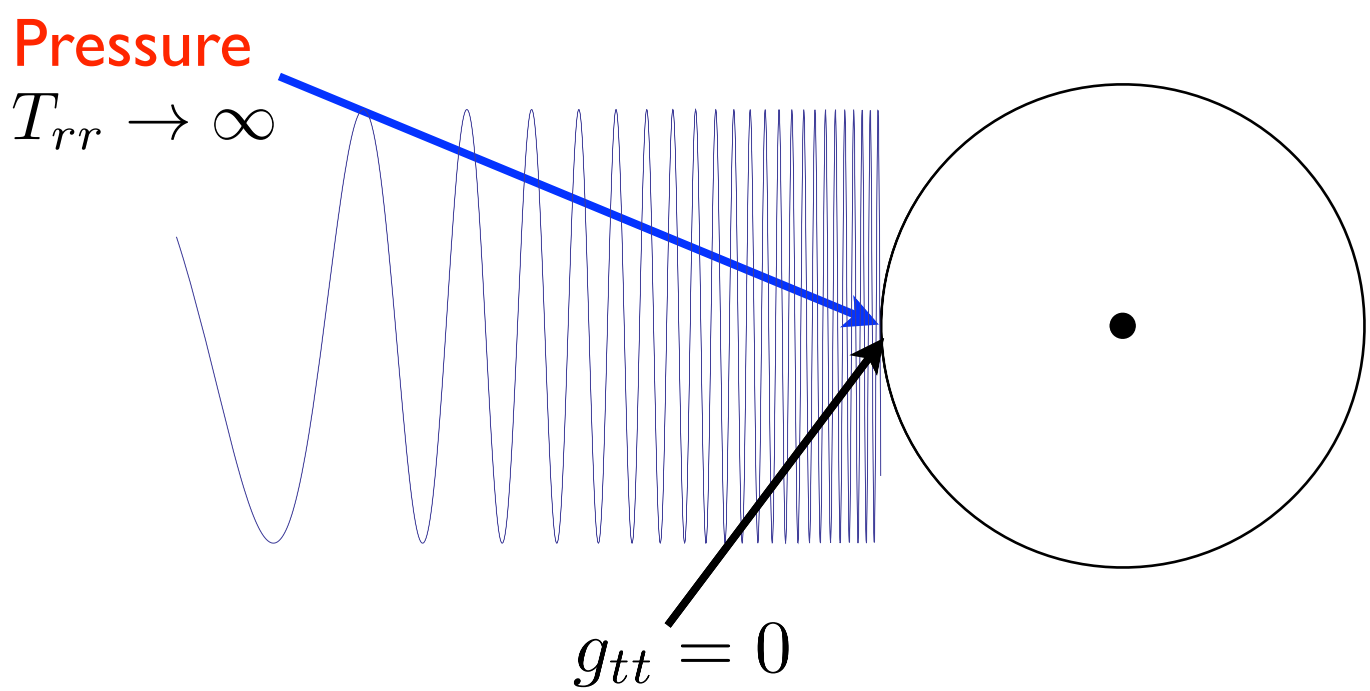
We again try to solve the wave equation, now in the Schwarzschild geometry



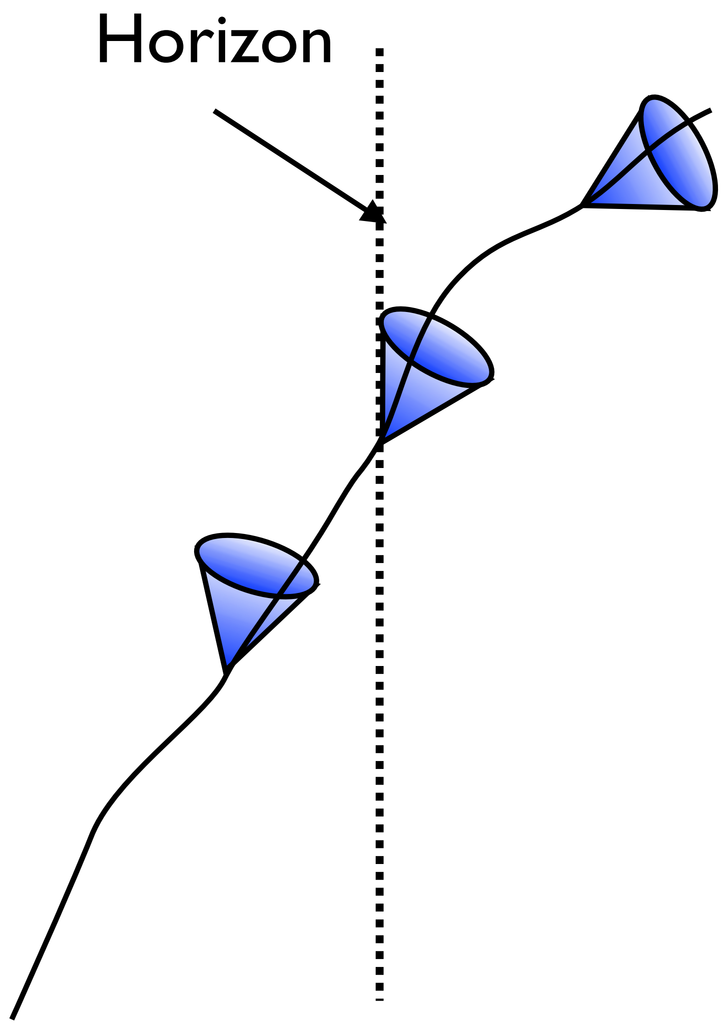
$$\Box\phi = 0 \quad , \qquad \phi = Y_{lm} f_{lmn}(r) e^{-i\omega t}$$

But this time we do not find good solutions $f_{lmn}(r)$

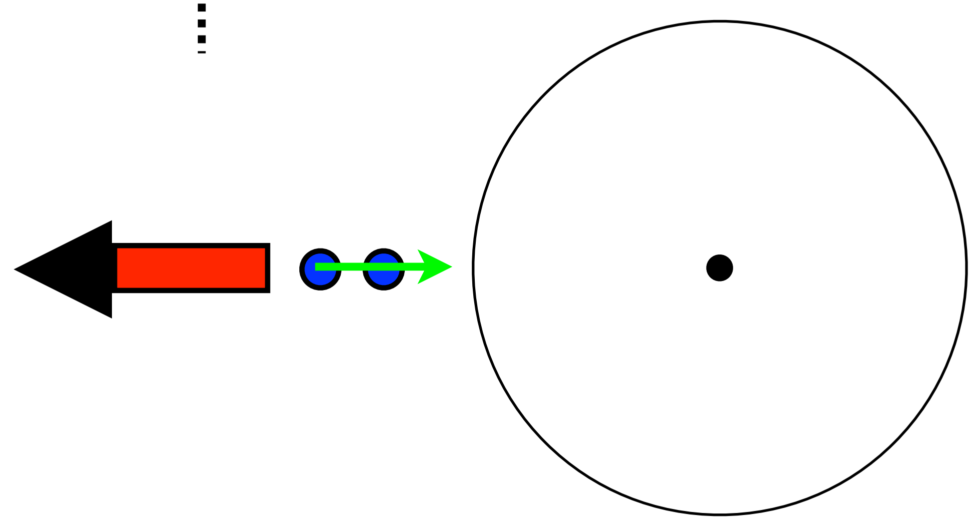
Field modes have divergent stress-energy



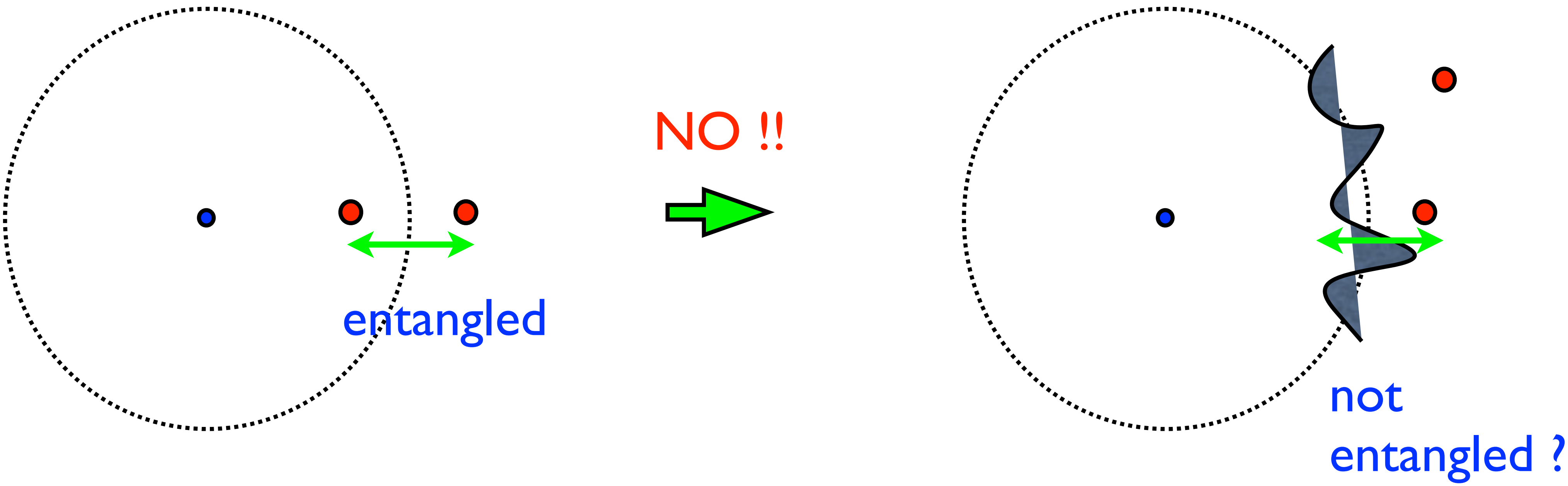
Why is it hard to find hair?



Large relative momentum needed
to keep the rocket stationary



Thus we see that we cannot deform the horizon (at either the classical or the quantum level) to change the production of entangled pairs



The information problem is really a combination of two things:

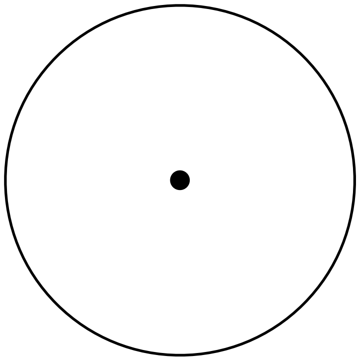
(a) Creation of entangled pairs by the Hawking process

(b) The 'no-hair' arguments which suggest that the state at the horizon cannot be changed

Q: Is there hair of any kind at the horizon? If so how ?

The structure of the black hole

Structure of the black hole



The black hole is described by the Schwarzschild metric

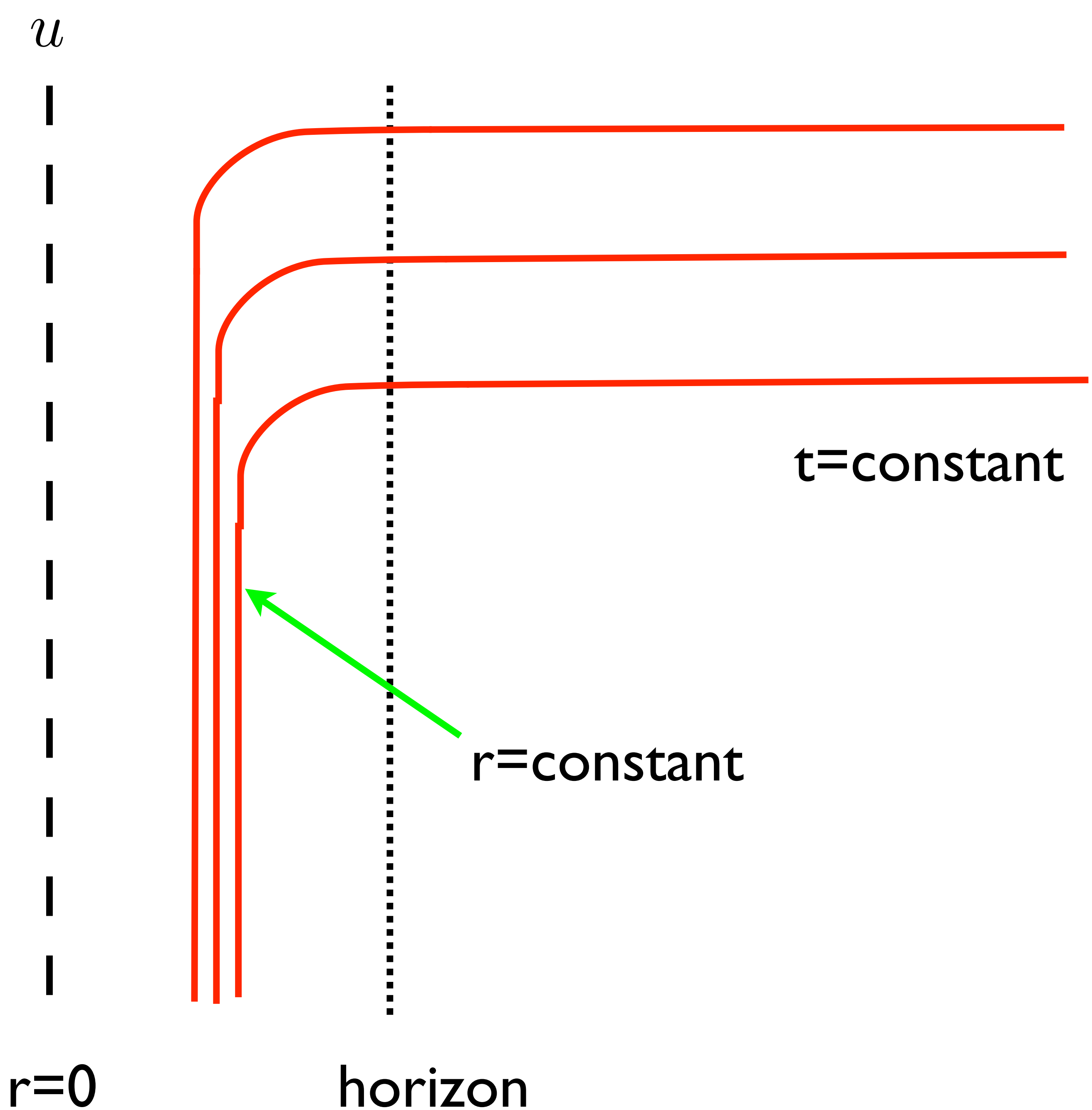
$$ds^2 \;=\; -(1-\frac{2M}{r})dt^2 \;+\; \frac{dr^2}{(1-\frac{2M}{r})} \;+\; r^2(d\theta^2+\sin^2\theta d\phi^2)$$

Crucial point about the black hole:

For $r > 2M$ the surface $t = constant$ is spacelike

For $r < 2M$ the surface $r = constant$ is spacelike

The spacelike slices in Eddington Finkelstein coordinates (u, r)



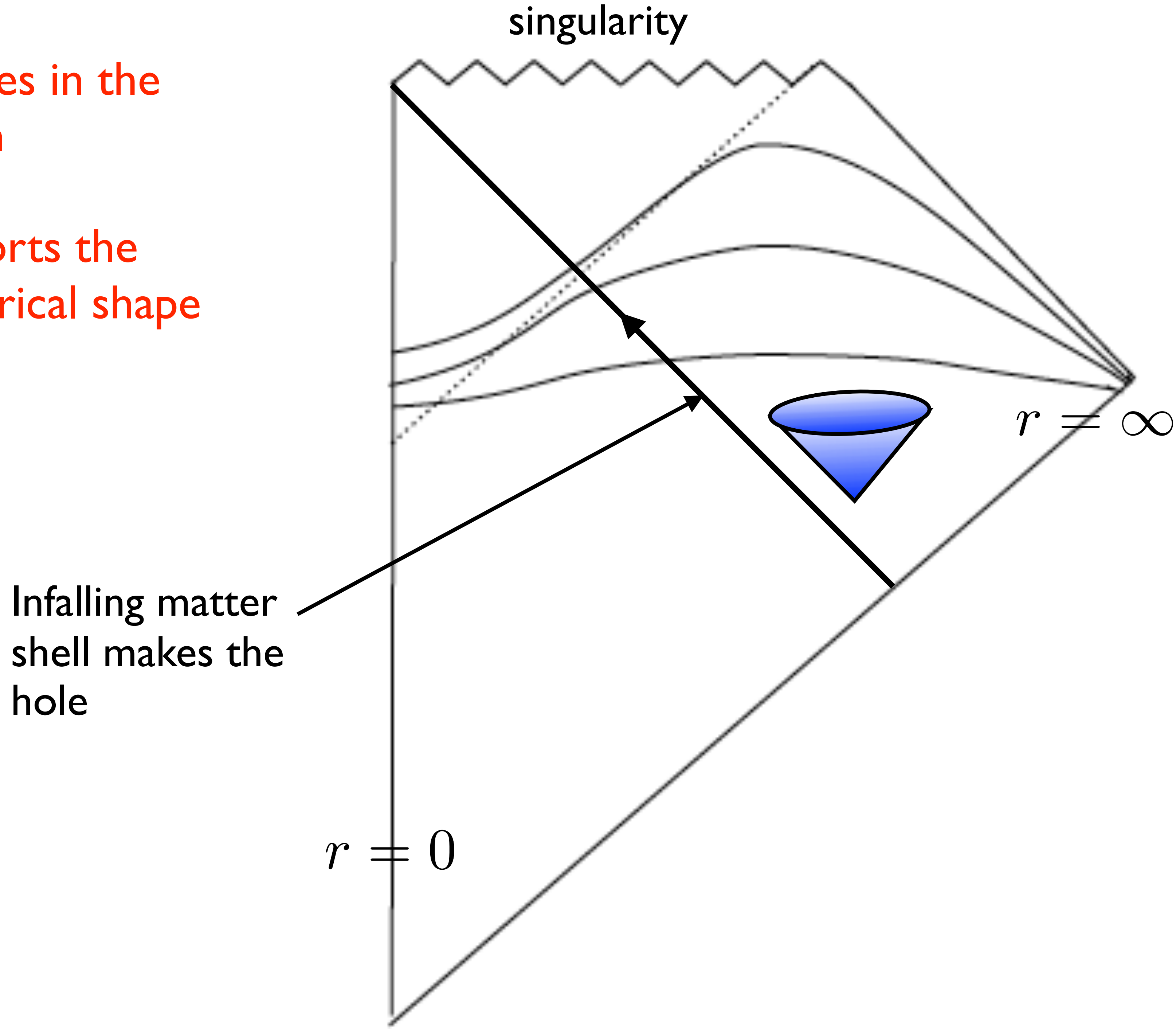
'later' slices are 'longer'

no time-independent slicing possible

Penrose diagram

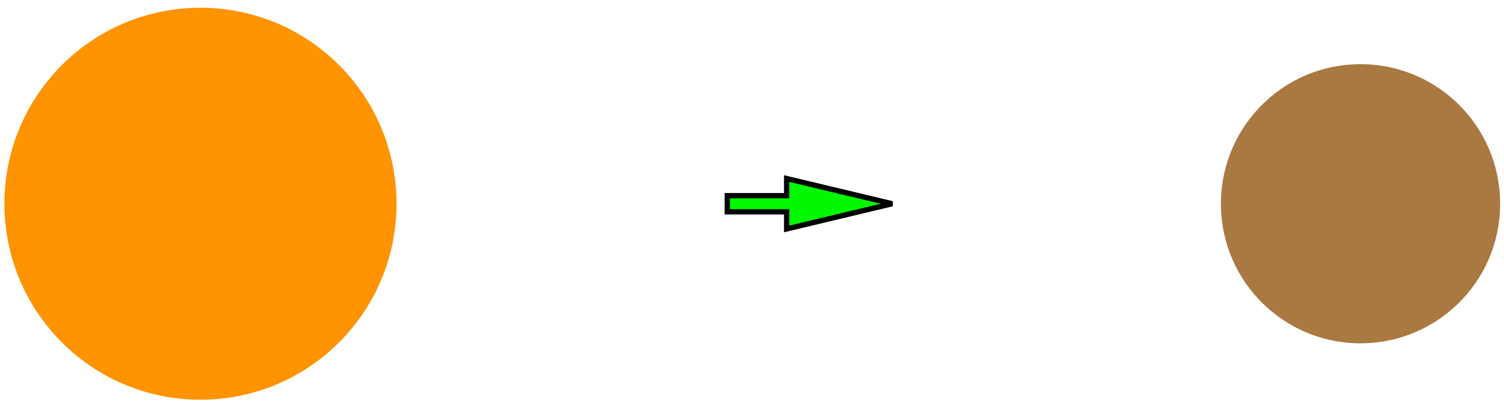
The smooth slices in the Penrose diagram

The scaling distorts the intrinsic geometrical shape of the slices



The stretching of slices leads to particle creation

In general, any change of the geometry leads to particle creation



$$\hat{\phi} = \sum_{l,m,n} \left(\hat{a}_{lmn} Y_{lm}(\theta,\phi) f_{lmn}(r) e^{-i\omega t} + \hat{a}^{\dagger}_{lmn} Y^{*}_{lm}(\theta,\phi) f^{*}_{lmn}(r) e^{i\omega t} \right)$$

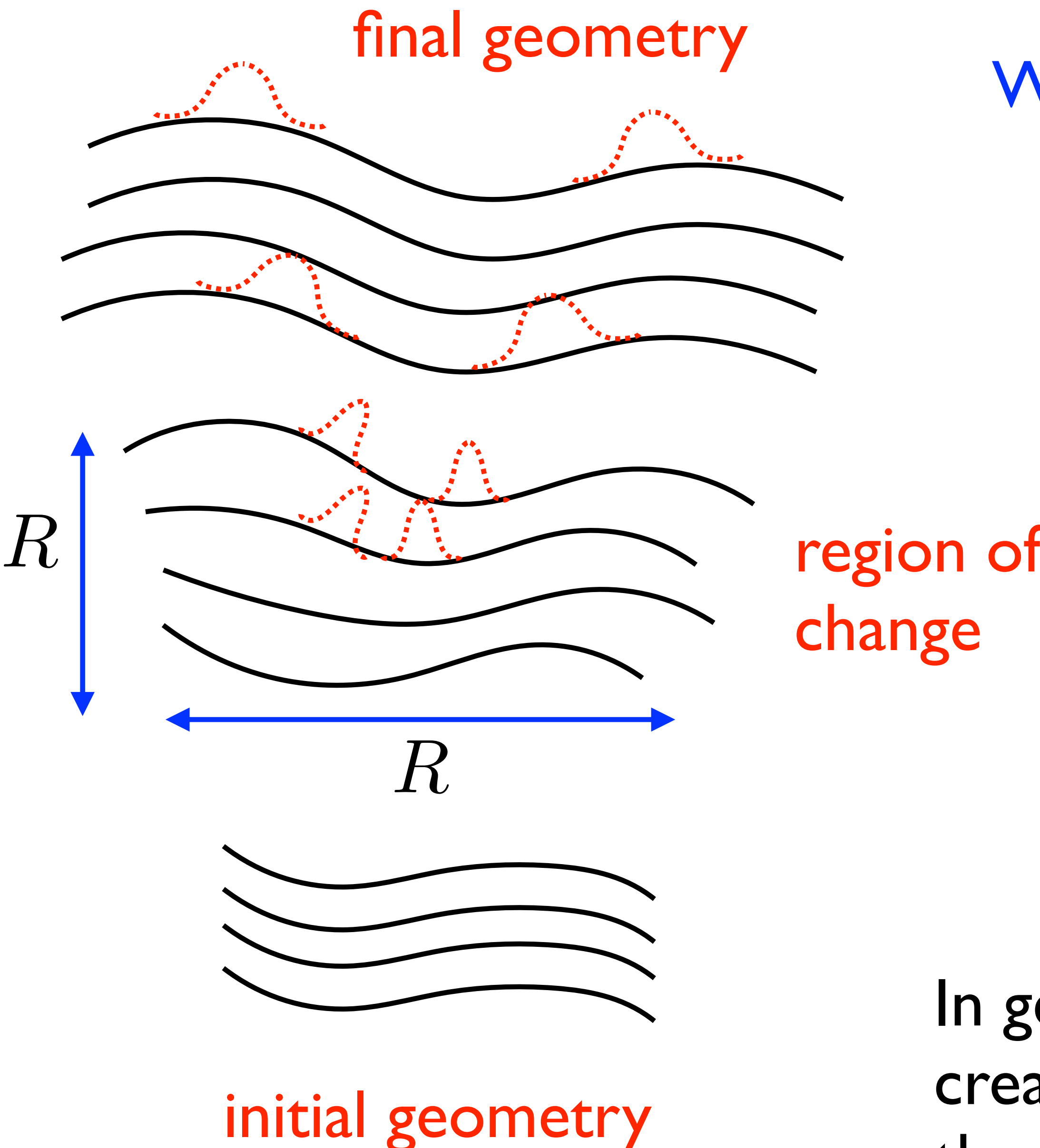
positive frequency solution to wave equation

negative frequency solution to wave equation

$$e^{-i\omega t} \rightarrow A e^{-i\omega' t} + B e^{i\omega' t}$$

Vacuum state gets changed to state containing particles

Scales



Number of created quanta ~ 1

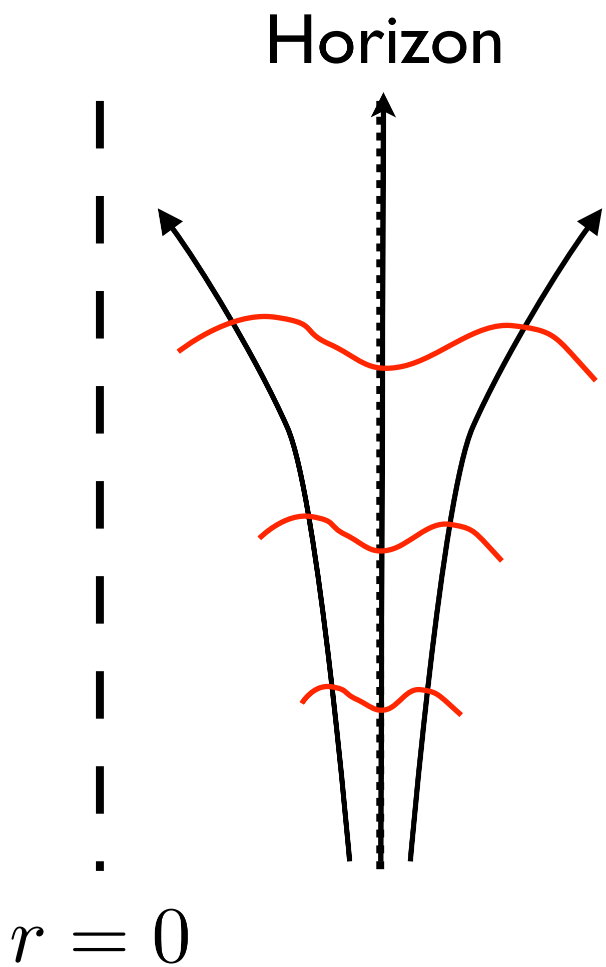
Wavelength of created quanta $\sim R$

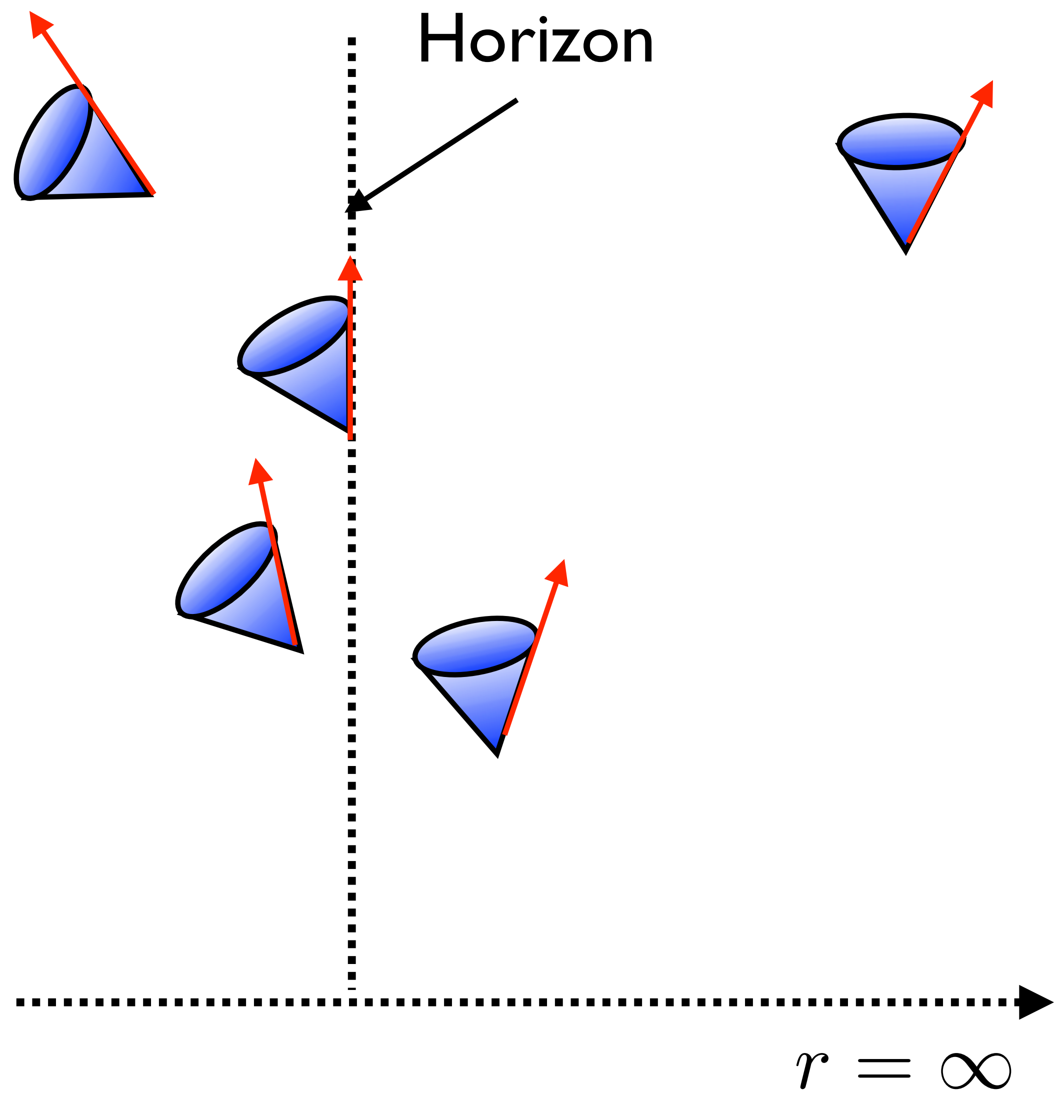
Typically in an entangled state

$$\frac{1}{\sqrt{2}}(00 + 11)$$

In general the energy of these created quanta is so low, that they create no significant deformation of the metric

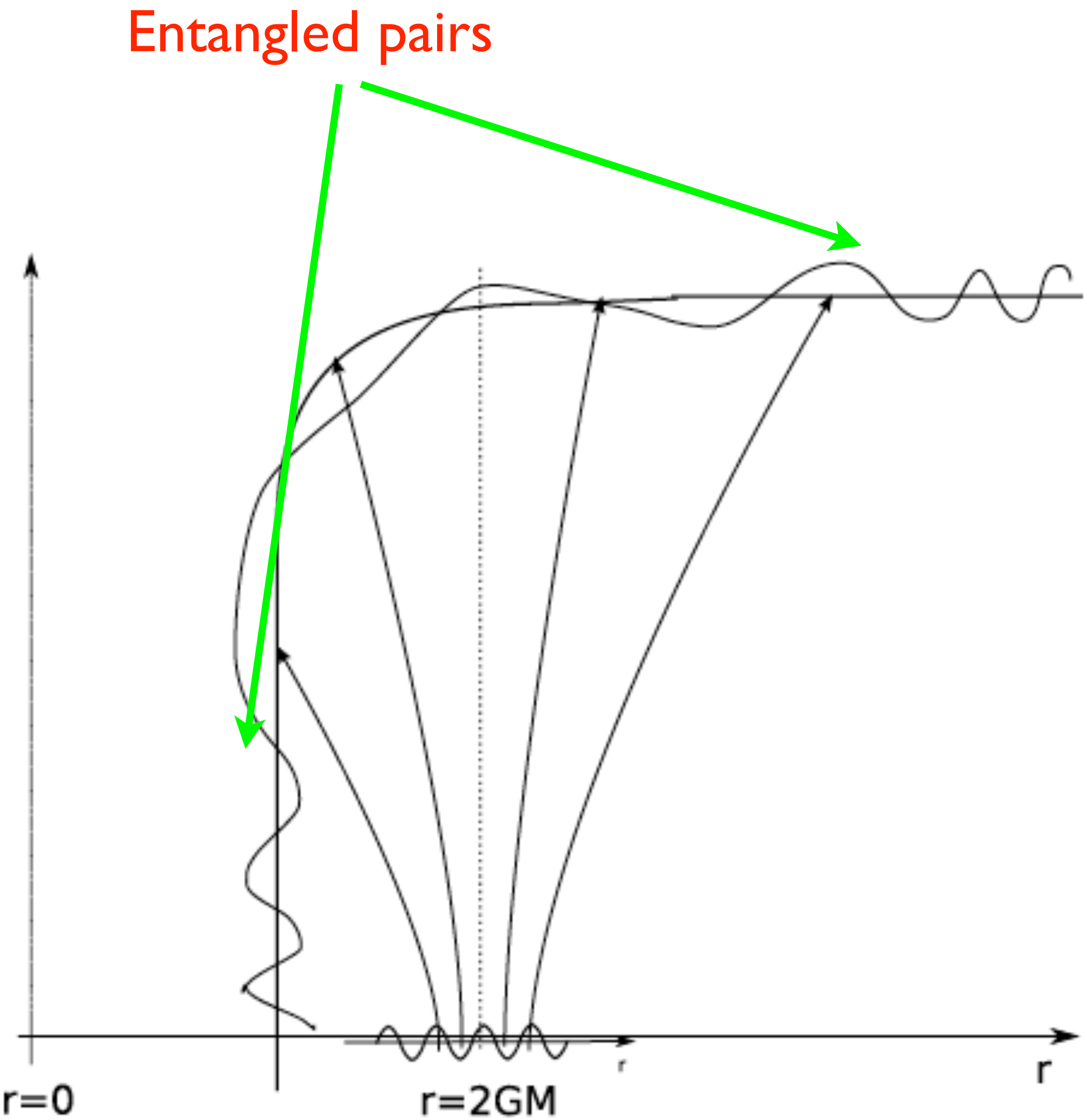
In the black hole, the geometry never stabilizes, so particle creation keeps going on …





Eikonal approximation: Constant phase along geodesics

The Hawking process



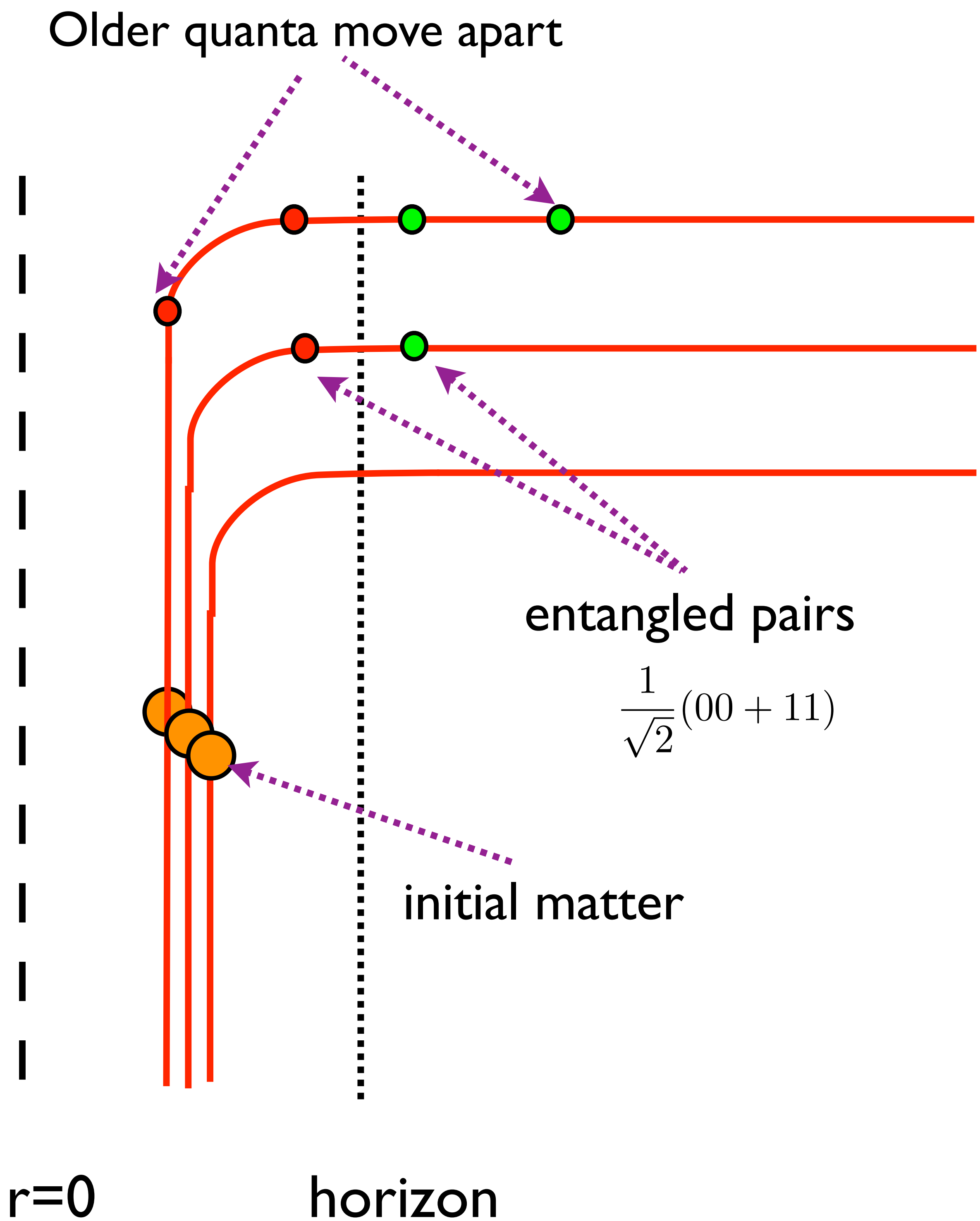
The fourier modes of a quantum field get distorted

If the initial mode was in a vacuum state, the new mode will typically NOT be in a vacuum state

Carrying out the details, we find that a pair of particles is created

There is a particle outside IF there is a particle inside

There is no particle outside IF there is no particle inside



Thus older quanta get 'flushed away' from the pair creation region

New pairs are then created again from the vacuum

Scales for Hawking radiation

There is only one length scale in the Schwarzschild metric

$$R = 2GM$$

The emitted quanta have wavelength $\lambda \sim R$

The time between emissions is $\Delta t \sim R$

The number of emitted quanta is

$$N \sim M/(1/R) \sim GM^2 \sim \left(\frac{M}{m_p}\right)^2$$

$$(\, G \sim l_p^2 \sim \frac{1}{m_p^2} \,)$$

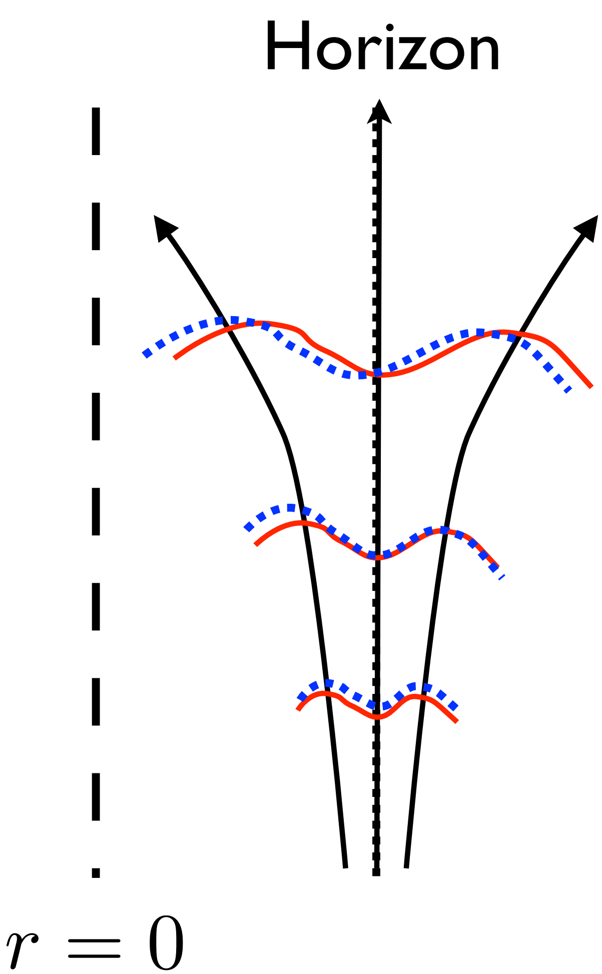
Q: Do we have the vacuum state at the horizon? If not, why not?

Q: If we have the vacuum state around the horizon, then do we get the creation of entangled pairs by stretching? If not, why not?

The possibility of small corrections

So what is the solution in string theory?

Some string theorists considered the possibility that small corrections could resolve the puzzle



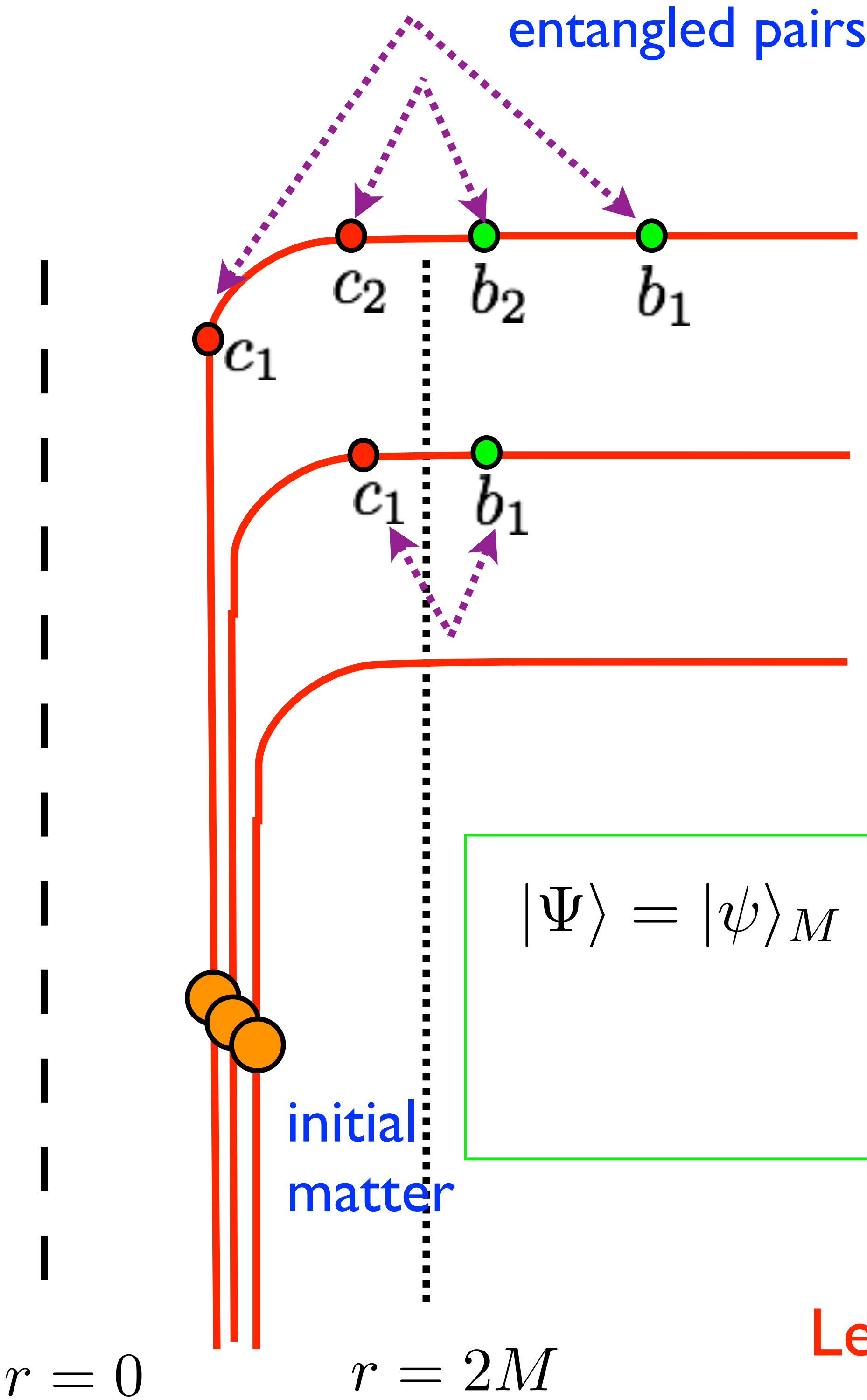
Very small corrections can of course come from any source

But why should we care?

The number of emitted quanta is very large

$$N \sim \left(\frac{M}{m_p}\right)^2 \gg 1$$

Maybe the net effect of all these small corrections is to remove the entanglement problem ...



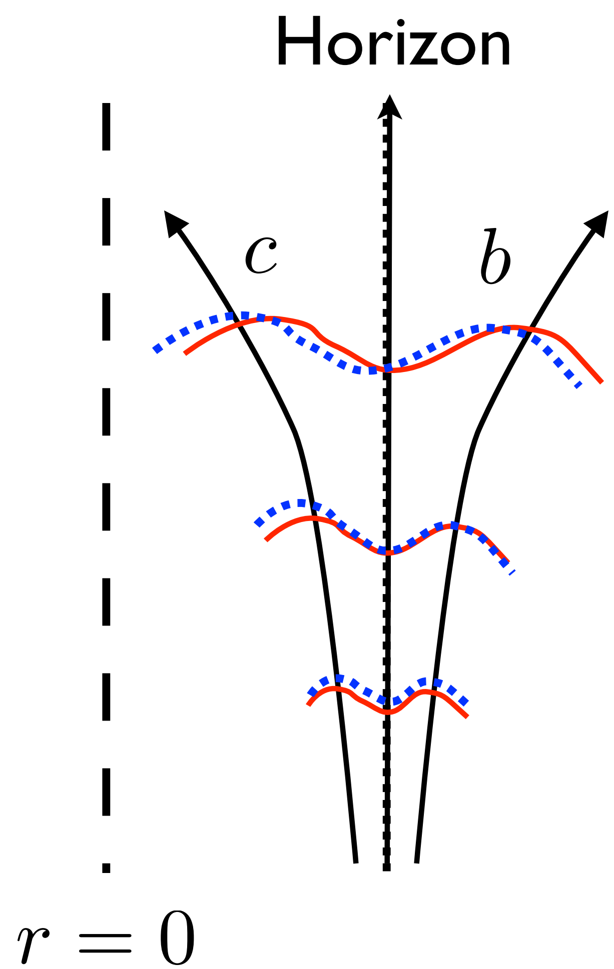
$$S_{ent} = \ln 2 + \ln 2 = 2 \ln 2$$

$$S_{ent} = \ln 2$$

$$S_{ent} = 0$$

$$\begin{aligned} |\Psi\rangle = |\psi\rangle_M \quad &\otimes \quad \left(\frac{1}{\sqrt{2}}|0\rangle_{c_1}|0\rangle_{b_1} + \frac{1}{\sqrt{2}}|1\rangle_{c_1}|1\rangle_{b_1}\right) \\ &\otimes \quad \left(\frac{1}{\sqrt{2}}|0\rangle_{c_2}|0\rangle_{b_2} + \frac{1}{\sqrt{2}}|1\rangle_{c_2}|1\rangle_{b_2}\right) \end{aligned}$$

Leading order Hawking computation



In Hawking's leading order calculation the entangled state had the schematic form

$$\frac{1}{\sqrt{2}}(|0\rangle_b|0\rangle_c + |1\rangle_b|1\rangle_c)$$

With the small corrections, we can get a small mixture of some other states

For simplicity we just take one orthogonal state

$$\frac{1}{\sqrt{2}}(|0\rangle_b|0\rangle_c + |1\rangle_b|1\rangle_c) + \epsilon_k \frac{1}{\sqrt{2}}(|0\rangle_b|0\rangle_c - |1\rangle_b|1\rangle_c)$$

$$(\epsilon_k \ll 1)$$

Suppose that at the first step of emission we have no change

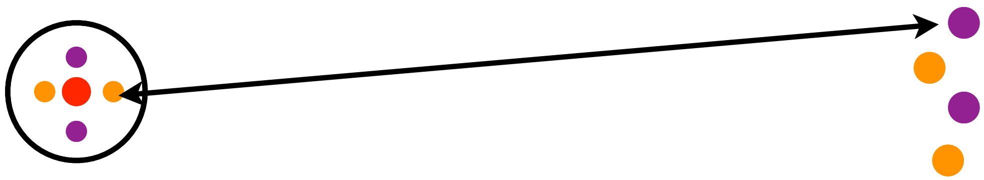
$$\frac{1}{\sqrt{2}}(|0\rangle_{b_1}|0\rangle_{c_1} + |1\rangle_{b_1}|1\rangle_{c_1})$$

At the second step of emission, suppose that if we had 00 at the first step then a 00 is slightly more likely, and if we had a 11 at the first step, then a 11 is slightly more likely

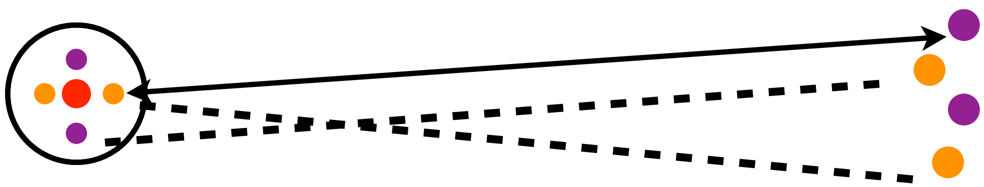
Overall state after two emissions

$$\frac{1}{2}\Big(|0\rangle_{b_1}|0\rangle_{c_1}[(1+\epsilon_1)|0\rangle_{b_2}|0\rangle_{c_2} + (1-\epsilon_1)|1\rangle_{b_2}|1\rangle_{c_2}]$$
$$+|1\rangle_{b_1}|1\rangle_{c_1}[(1+\epsilon_1')|0\rangle_{b_2}|0\rangle_{c_2} + (1-\epsilon_1')|1\rangle_{b_2}|1\rangle_{c_2}]\Big)$$

After N steps of emission, there are $\sim 2^N$ correction terms

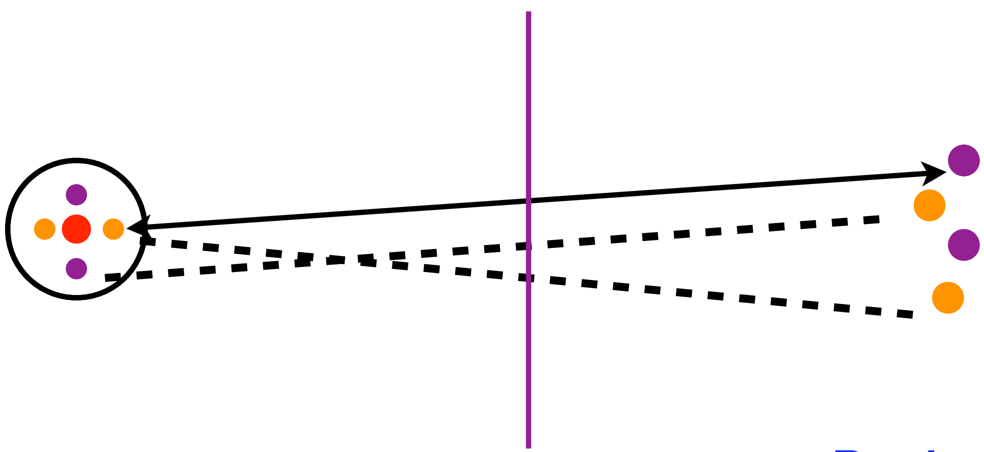


leading order



leading order +
subleading effects

Number of emitted quanta is very large $\sim (M/m_p)^2$

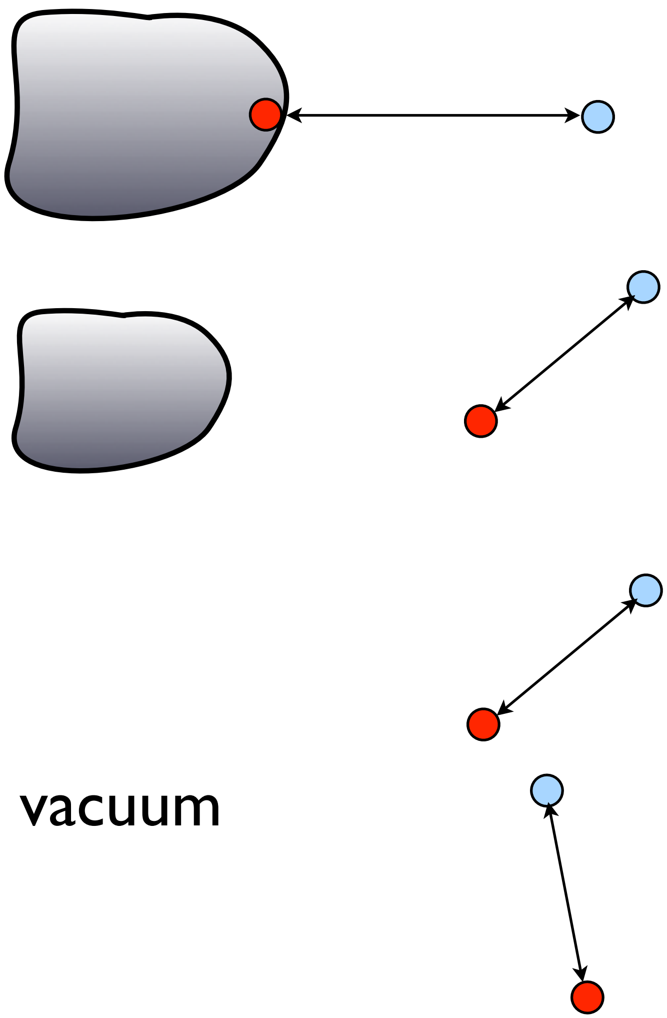


Perhaps with all these corrections,
the entanglement goes down to zero …

If the problem can be resolved this way then there was no ‘information paradox’ in the first place:

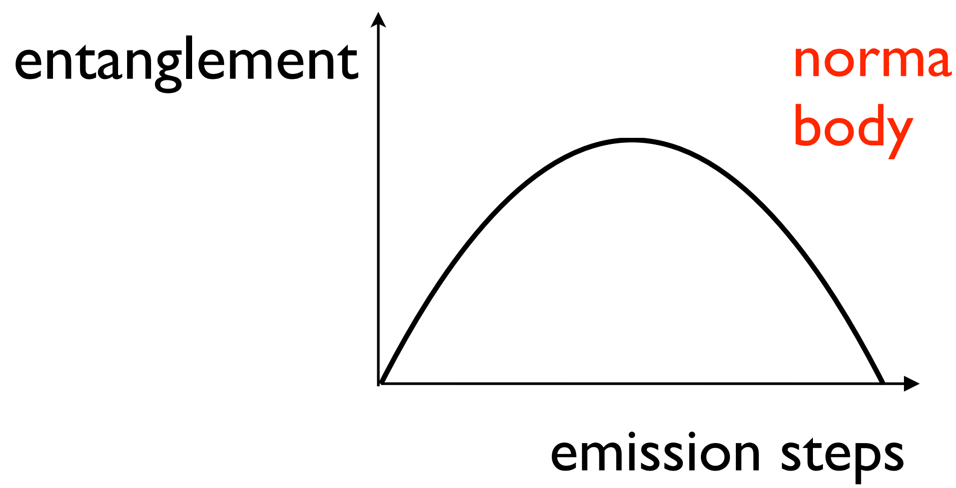
We would say that Hawking did a leading order calculation, and when subleading terms are taken into account, the problematic entanglement disappears

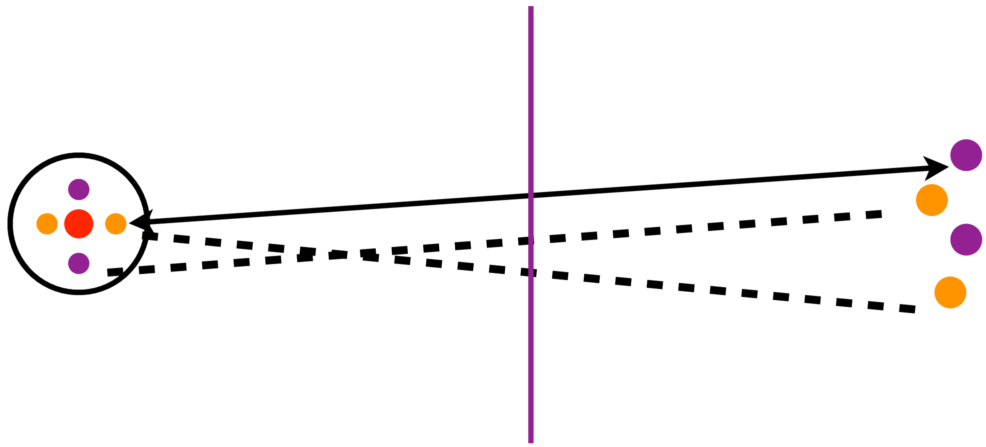
This possibility looked plausible given how normal bodies behave …



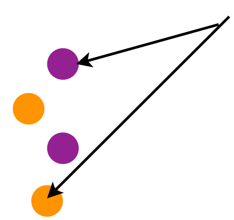
Emitted photon will be entangled with the atom which emitted it

But after some time the atom floats out like ash, so only things at infinity are entangled with each other

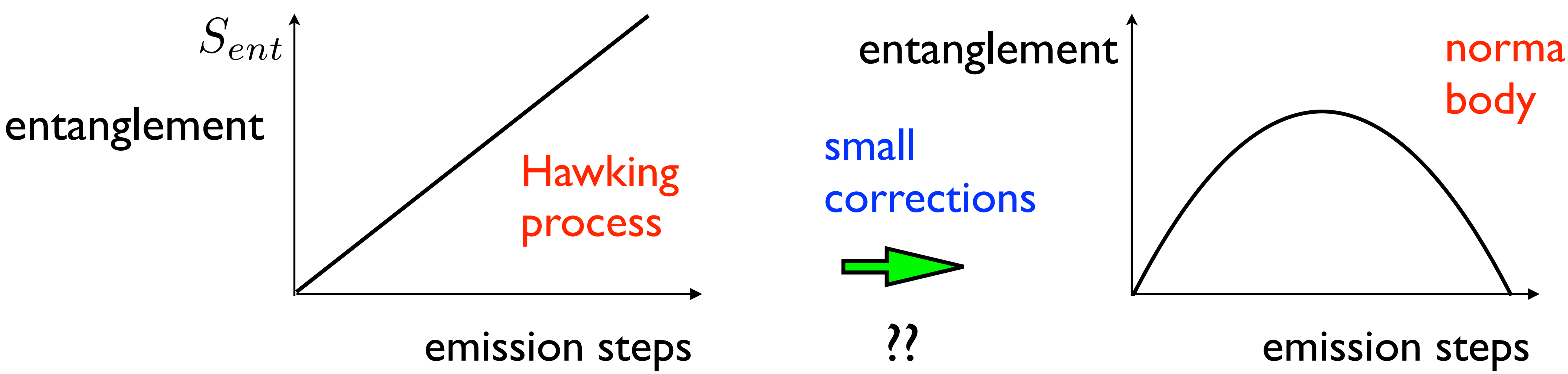




These small corrections may disentangle the radiation from the remnant



vacuum

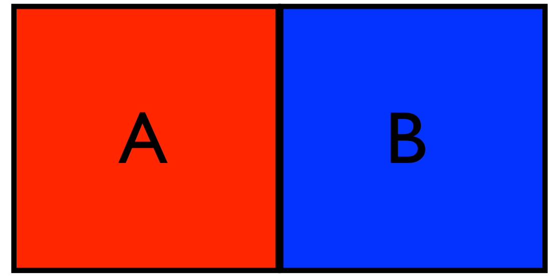


The small corrections theorem

(SDM arXiv:0909.1038)

Quantum entanglement entropy

Suppose systems A and B
are entangled with each other



$$|\Psi\rangle = \sum_{i=1}^{N} C_{mn}\ \psi_m^A \otimes \chi_n^B$$

We can choose an orthonormal basis for A and an orthonormal basis of B such that

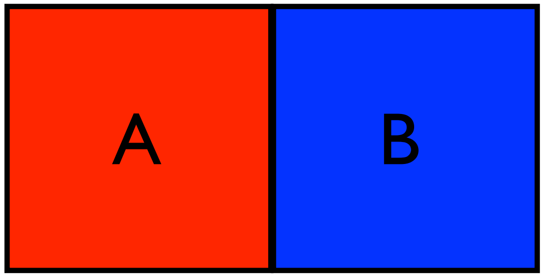
$$|\Psi\rangle = \sum_{i=1}^{M} C_i\ \psi_i^A \otimes \chi_i^B$$

We can trace out the degrees of freedom in B, to get a reduced density matrix describing A

$$\rho_A = \sum_{i=1}^{M} |C_i|^2\ |\psi_i\rangle\langle\psi_i|$$

The entanglement entropy of A with the rest of the system (B) is

$$S(A) = -Tr[\rho_A \ln \rho_A]$$
$$= \sum_{i=1}^{M} |C_i|^2 \ln |C_i|^2$$

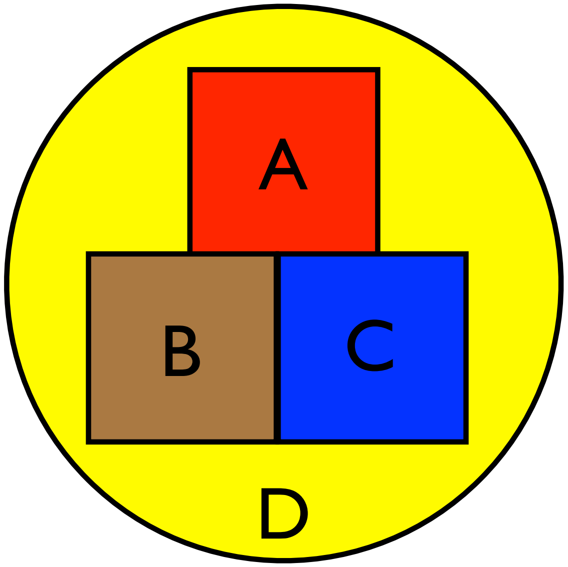


If A is entangled with B, then B is entangled with A

$$S(A) = S(B)$$

Quantum entropy can behave very differently from classical entropy !

$$S(A+B) = 0$$



We can have many subsystems entangled with each other

$$S(A) = -Tr[\rho_A \ln \rho_A]$$

Entanglement entropy of A with rest of system (B+C+D)

$$S(A+B) = -Tr[\rho_{A+B} \ln \rho_{A+B}]$$

Entanglement entropy of A+B with rest of system (C+D)

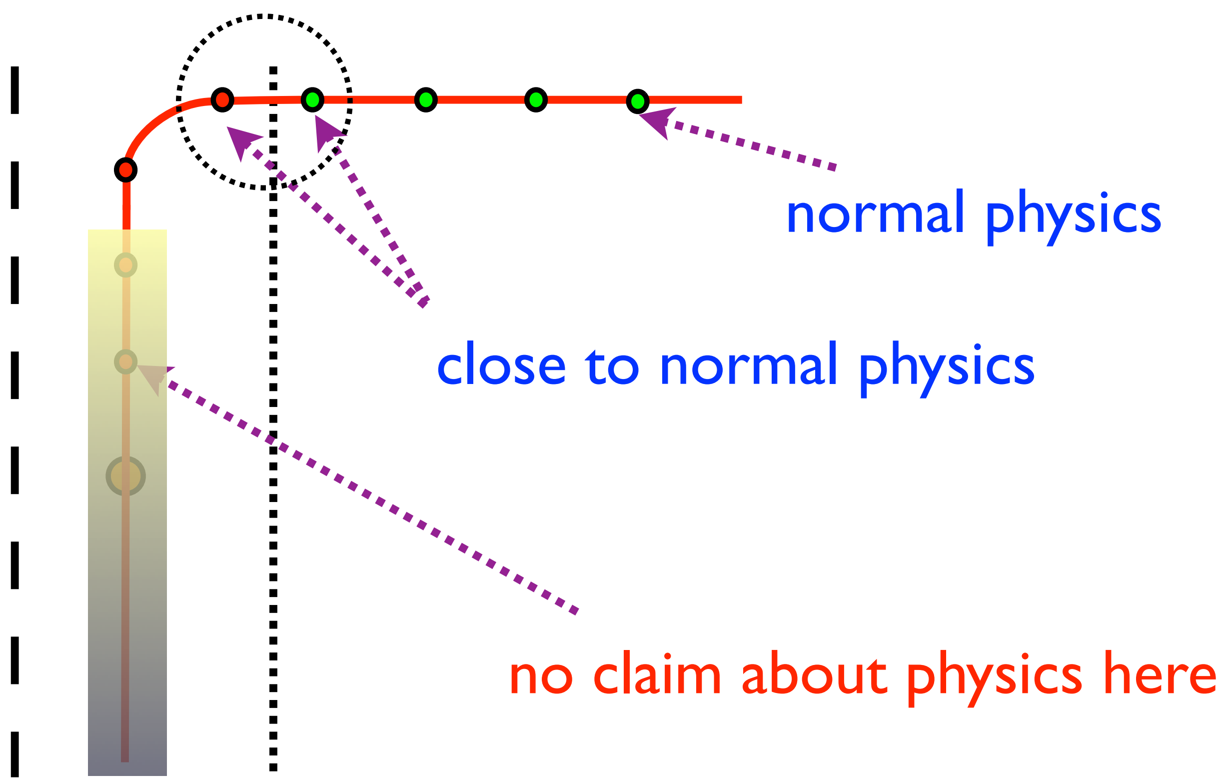
Since overall system has a pure state, $S(A+B) = S(C+D)$ etc.

Strong subadditivity $$S(A+B) + S(B+C) \geq S(A) + S(C)$$

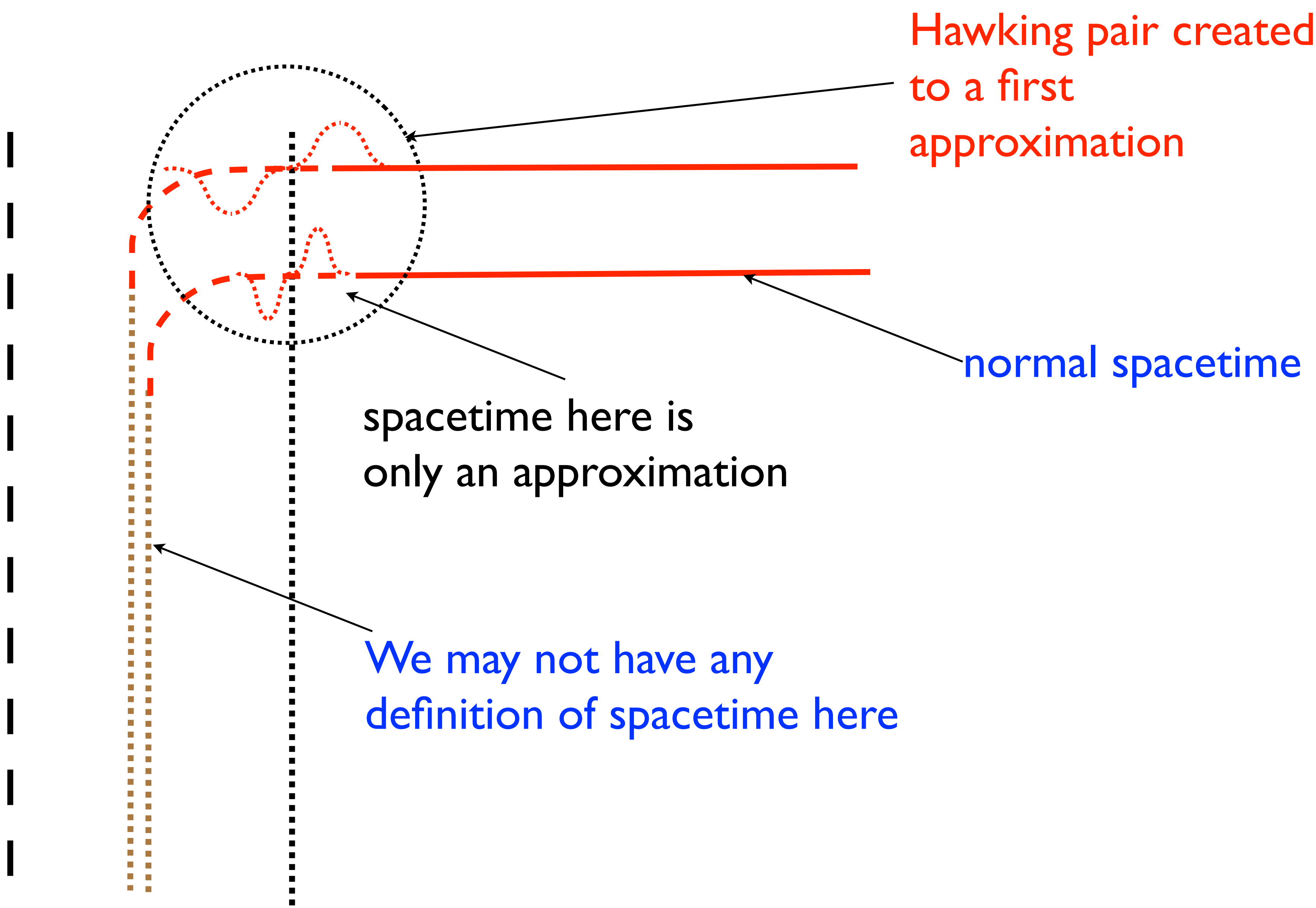
We make the following model:

(a) Outside the black hole ($r > 10M$) we have normal physics of quanta

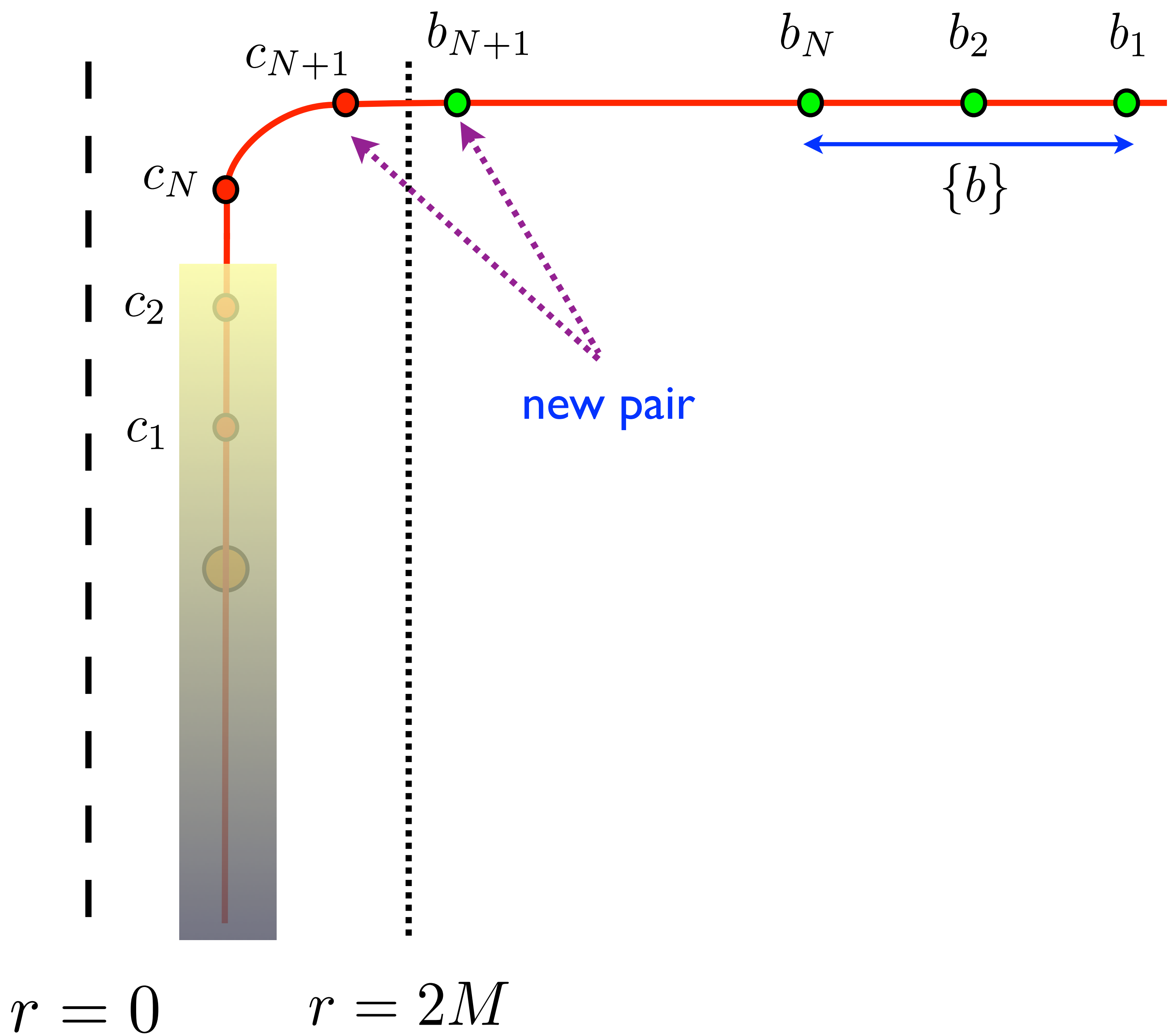
(b) The region $r < 10M$ can be described exactly by some dual field theory. But the semiclassical black hole physics is recovered to a good approximation for low energy processes over short times



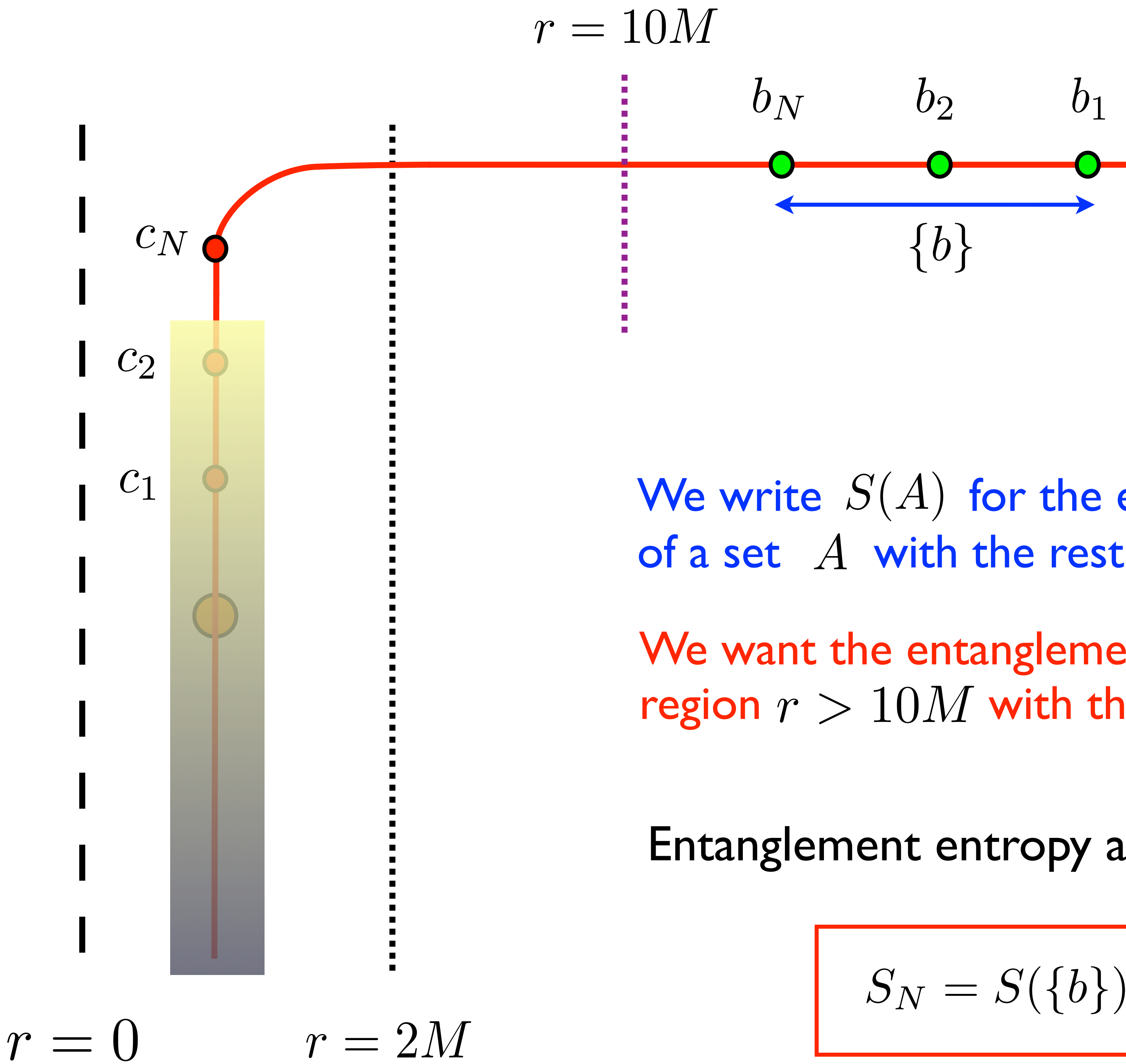
(c) Evolution near the horizon would create one pair to a first approximation



(d) Notation:



(e) Entropy at step N



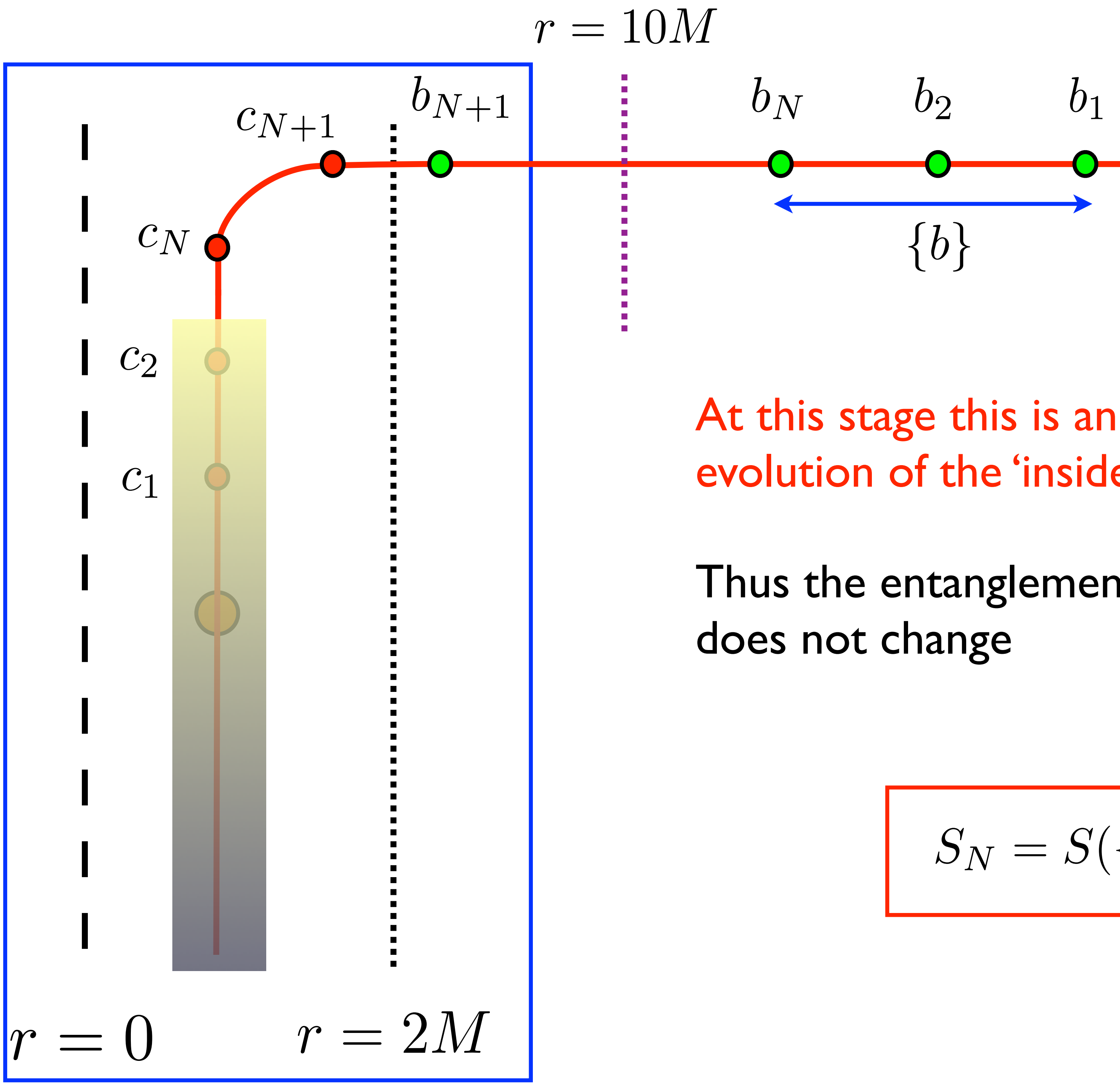
We write $S(A)$ for the entanglement entropy of a set A with the rest of the system

We want the entanglement of the outside region $r > 10M$ with the inside

Entanglement entropy at step N is

$$\boxed{S_N = S(\{b\})}$$

(f) Creation of a new pair



At this stage this is an unitary internal evolution of the 'inside' system.

Thus the entanglement with the outside does not change

$$\boxed{S_N = S(\{b\})}$$

(f) Entanglements of the new pair

In the leading order Hawking process, the state of the newly created state of the pair is

$$|\Psi\rangle_{pair} = \frac{1}{\sqrt{2}}(|0\rangle_{b_{N+1}}|0\rangle_{c_{N+1}} + |1\rangle_{b_{N+1}}|1\rangle_{c_{N+1}})$$

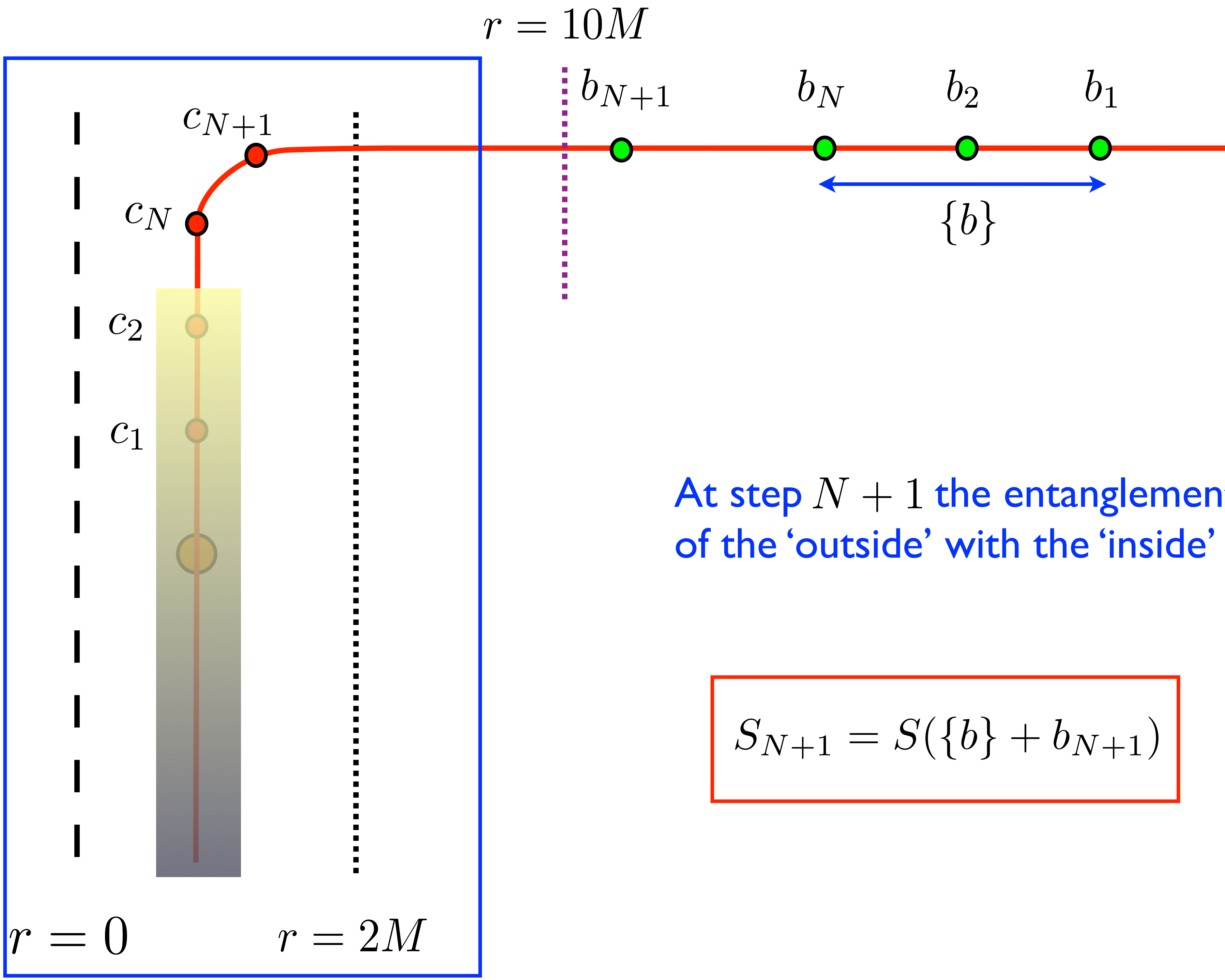
Thus b_{N+1} is maximally entangled with c_{N+1}, and the set b_{N+1}, c_{N+1} is not entangled with anything else

$$S(b_{N+1}, c_{N+1}) = 0 \quad , \qquad S(b_{N+1}) = \ln 2$$

Since the actual evolution has to approximate the leading order evolution, we require

$$\boxed{\begin{array}{l} S(b_{N+1}, c_{N+1}) < \epsilon_1 \\ \epsilon_1 \ll 1 \end{array}} \qquad \boxed{\begin{array}{l} S(c_{N+1}) > \ln 2 - \epsilon_2 \\ \epsilon_2 \ll 1 \end{array}}$$

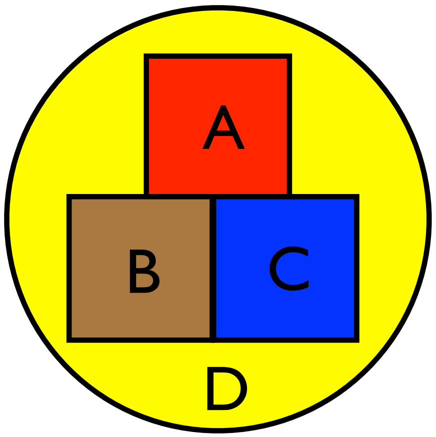
(g) Entanglement entropy at step $N+1$



At step $N+1$ the entanglement of the 'outside' with the 'inside' is

$$\boxed{S_{N+1} = S(\{b\} + b_{N+1})}$$

(h) The strong subadditivity of quantum entanglement entropy



$$S(A+B) + S(B+C) \geq S(A) + S(C)$$

We take

$$A = \{b\}\ , \qquad B = b_{N+1}\ , \qquad C = c_{N+1}$$

We recall $\quad S_N = S(\{b\})\ , \qquad S_{N+1} = S(\{b\} + b_{N+1})$

$$S(b_{N+1}, c_{N+1}) < \epsilon_1\ , \qquad S(c_{N+1}) > \ln 2 - \epsilon_2$$

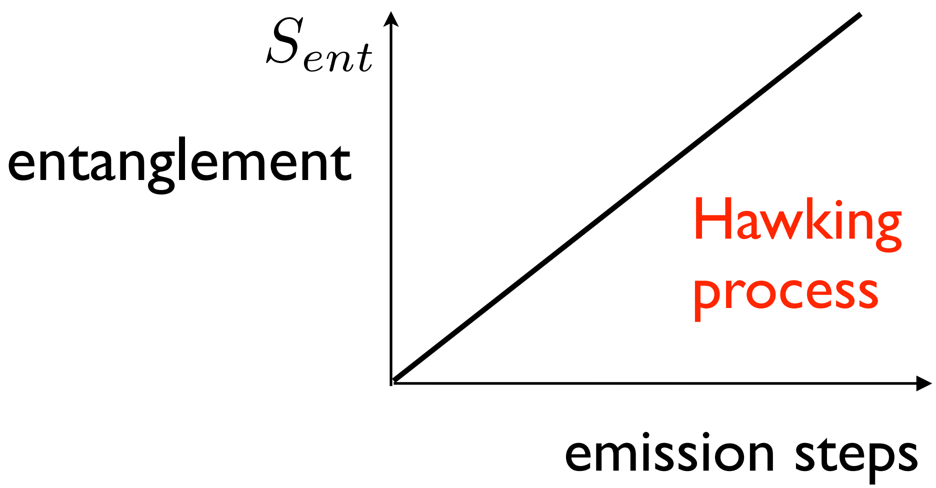
Then we get

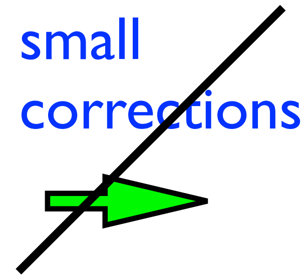
$$S(\{b\} + b_{N+1}) + S(b_{N+1} + c_{N+1}) \geq S(\{b\} + S(c_{N+1})$$

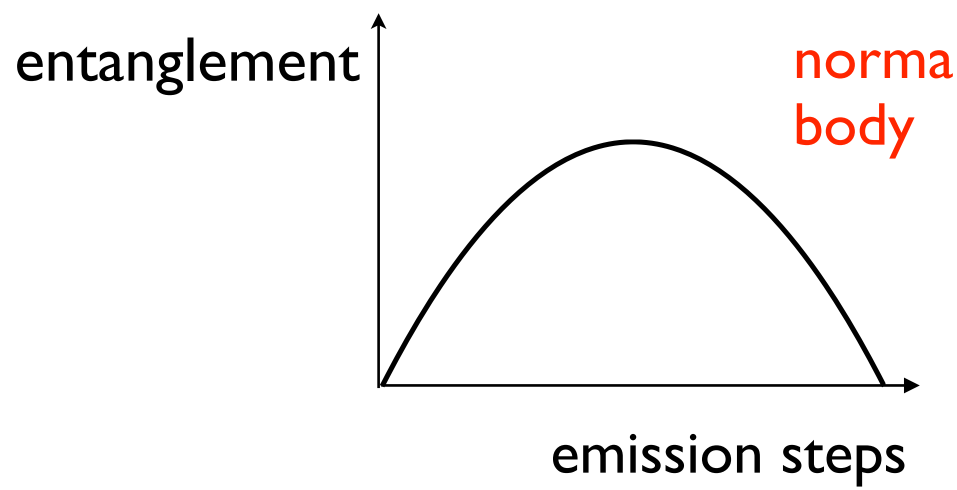
$$\boxed{S_{N+1} > S_N + \ln 2 - (\epsilon_1 + \epsilon_2)}$$

(SDM 2009)

Thus we see that







Thus the Hawking argument is stable against small corrections