The Basic Structure of Scattering Amplitudes

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(these slides are supposed to be integrated with the blackboard notes)

References:

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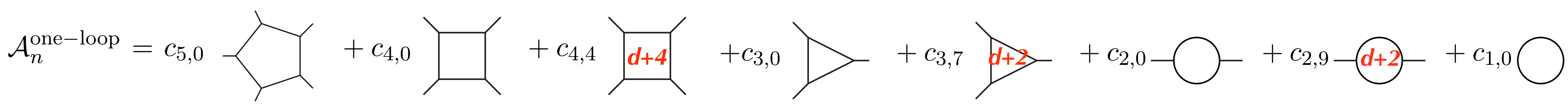
- Badger, Frellersvig, Zhang, *Hepta-cuts of Two-Loop Scattering Amplitudes,* 1202.2019 [hep-ph]
- Zhang, *Integrand-level Reduction of Loop Amplitudes by Computational Algebraic Geometry Methods* 1205.5707 [hep-ph]
- Mirabella, Peraro, Ossola, & P.M., *Scattering Amplitudes from Multivariate Polynomial Division,* 1205.7087 [hep-ph]

One-Loop Integrand Decomposition

Ossola, Papadopoulos, Pittau
Ellis, Giele, Kunszt, Melnikov

$$\mathcal{A}_n^{\rm one-loop} = \int d^{-2\epsilon}\mu \int d^4 q \; A_n(q,\mu^2) \,, \qquad A_n(q,\mu^2) \equiv \frac{\mathcal{N}_n(q,\mu^2)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{n-1}} \qquad \bar{D}_i = (\bar{q}+p_i)^2 - m_i^2 = (q+p_i)^2 - m_i^2 - \mu^2$$

We use a bar to denote objects living in $d = 4-2\epsilon$ dimensions $\quad \bar{\not q} = \not q + \not \mu \,, \quad$ with $\quad \bar{q}^2 = q^2 - \mu^2 \,.$



Passarino, Veltman; Tarasov

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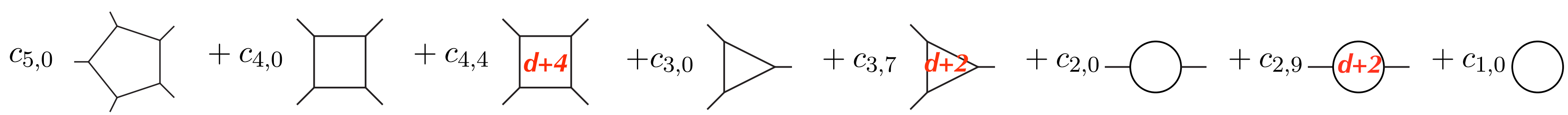
$\mathcal{A}_n^{\rm one-loop} =$



Passarino, Veltman; Tarasov

☑ @ the integrand-level

$$A_n(q,\mu^2) \neq \frac{c_{5,0}}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3\bar{D}_4} + \frac{c_{4,0}+c_{4,4}\mu^4}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3} + \frac{c_{3,0}+c_{3,7}\mu^2}{\bar{D}_0\bar{D}_1\bar{D}_2} + \frac{c_{2,0}+c_{2,9}\mu^2}{\bar{D}_0\bar{D}_1} + \frac{c_{1,0}}{\bar{D}_0}$$

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Ossola, Papadopoulos, Pittau
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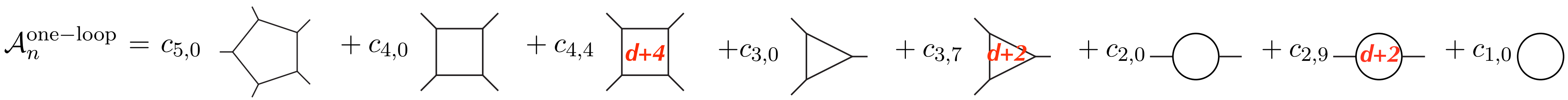


Passarino, Veltman; Tarasov

@ the integrand-level

$$A_n(q,\mu^2) \neq \frac{c_{5,0}}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3\bar{D}_4} + \frac{c_{4,0}+c_{4,4}\mu^4}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3} + \frac{c_{3,0}+c_{3,7}\mu^2}{\bar{D}_0\bar{D}_1\bar{D}_2} + \frac{c_{2,0}+c_{2,9}\mu^2}{\bar{D}_0\bar{D}_1} + \frac{c_{1,0}}{\bar{D}_0}$$

$$= \frac{c_{5,0}+f_{01234}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3\bar{D}_4} + \frac{c_{4,0}+c_{4,4}\mu^4+f_{0123}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3} + \frac{c_{3,0}+c_{3,7}\mu^2+f_{012}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2} + \frac{c_{2,0}+c_{2,9}\mu^2+f_{01}(q,\mu^2)}{\bar{D}_0\bar{D}_1} + \frac{c_{1,0}+f_0(q,\mu^2)}{\bar{D}_0}$$

Spurious Terms

$$\int d^{-2\epsilon}\mu \int d^4 q \; \frac{f_{i_1 i_2 \cdots i_n}(q,\mu^2)}{\bar{D}_{i_1}\bar{D}_{i_2}\cdots\bar{D}_{i_n}} = 0\,.$$

Ossola, Papadopoulos, Pittau

Parametric form of the residues: known

Multi-(particle)-pole decomposition

$$A(\bar{q}) = \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} + \sum_{i<<\ell}^{n-1} \frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell} + \sum_{i<<k}^{n-1} \frac{\Delta_{ijk}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k} + \sum_{i<j}^{n-1} \frac{\Delta_{ij}(\bar{q})}{\bar{D}_i\bar{D}_j} + \sum_{i}^{n-1} \frac{\Delta_{i}(\bar{q})}{\bar{D}_i},$$

Integrand Reduction Formula

$$N(\bar{q}) = \sum_{i<<m}^{n-1} \Delta_{ijk\ell m}(\bar{q}) \prod_{h\neq i,j,k,\ell,m}^{n-1} \bar{D}_h + \sum_{i<<\ell}^{n-1} \Delta_{ijk\ell}(\bar{q}) \prod_{h\neq i,j,k,\ell}^{n-1} \bar{D}_h + \sum_{i<<k}^{n-1} \Delta_{ijk}(\bar{q}) \prod_{h\neq i,j,k}^{n-1} \bar{D}_h + \sum_{i<j}^{n-1} \Delta_{ij}(\bar{q}) \prod_{h\neq i,j}^{n-1} \bar{D}_h + \sum_{i}^{n-1} \Delta_{i}(\bar{q}) \prod_{h\neq i}^{n-1} \bar{D}_h,$$

Use your favourite generator,
(for Feynman Diagrams, or for products of tree-amplitudes),
and sample N(q) as many time as the number of unknown coefficients

Cuts and Residues

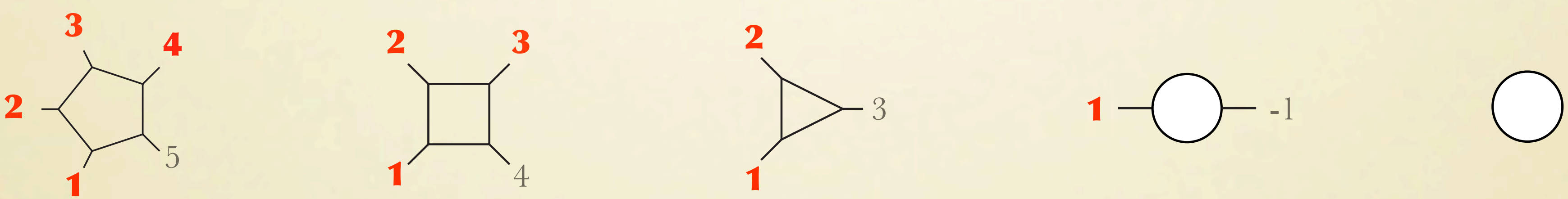
cut-associated basis

For each cut $(ijk\cdots)$, $D_i = D_j = D_k = \cdots = 0$, a basis of four massless vectors

$$\left\{e_1^{(ijk\cdots)}, e_2^{(ijk\cdots)}, e_2^{(ijk\cdots)}, e_4^{(ijk\cdots)}\right\}$$

$$\left(e_i^{(ijk\cdots)}\right)^2 = 0\ , \qquad e_1^{(ijk\cdots)}\cdot e_3^{(ijk\cdots)} = e_1^{(ijk\cdots)}\cdot e_4^{(ijk\cdots)} = 0\ ,$$
$$e_2^{(ijk\cdots)}\cdot e_3^{(ijk\cdots)} = e_2^{(ijk\cdots)}\cdot e_4^{(ijk\cdots)} = 0\ , \qquad e_1^{(ijk\cdots)}\cdot e_2^{(ijk\cdots)} = -e_3^{(ijk\cdots)}\cdot e_4^{(ijk\cdots)} = 1$$

use independent external momenta + auxiliary orthogonal complement:



4-vectors vs components

- Loop momentum decomposition

$$q + p_i = \sum_{\alpha=1}^{4} x_\alpha\ e_\alpha^{(ijk\cdots)}$$

The Shape of Residues

legs	basis	
	external (p_i)	auxiliary (v_i)
5	4	0
4	3	1
3	2	2
2	1	3
1	0	4

Δ-variables

- ISP's = Irreducible Scalar Products:
  - q-components which can variate under cut-conditions
  - spurious: vanishing upon integration
  - non-spurious: non-vanishing upon integration $\Rightarrow$ MI's
- @ 1-Loop
  - $(q \cdot p_i)$ are ALL reducible
  - ISP's could be chosen to be ALL spurious — Pittau, de l'Aguila
  - n-ple cut identifies an n-point diagram

Integrand-Reduction beyond One-Loop

Ossola & P.M. (2011)

Badger, Frellesvig, Zhang (2011)

Zhang (2012)

Mirabella, Ossola, Peraro, & P.M (2012)

Kleiss, Malamos, Papadopoulos, Verheynen (2012)

Two-Loop Integrand Reduction

Ossola & P.M. (2011)

- Four-Dimensional Algorithm

$$\mathcal{A}_n = \int d^{4-2\epsilon}q \int d^{4-2\epsilon}k \; A(q,k) \; , \qquad A(q,k) = \frac{N(q,k)}{D_1 D_1 \cdots D_n} \; , \qquad D_i = (\alpha_i q + \beta_i k + p_i)^2 - m_i^2, \qquad \alpha_i, \beta_i \in \{0,1\}$$

- educated guess: Master-Decomposition Formula (4-dim)

$$A(q,k) = \sum_{i_1 << i_8}^{n} \frac{\Delta_{i_1,\ldots,i_8}(q,k)}{D_{i_1} D_{i_2} \ldots D_{i_8}} + \sum_{i_1 << i_7}^{n} \frac{\Delta_{i_1,\ldots,i_7}(q,k)}{D_{i_1} D_{i_2} \ldots D_{i_7}} + \ldots + \sum_{i_1 << i_2}^{n} \frac{\Delta_{i_1,i_2}(q,k)}{D_{i_1} D_{i_2}} \, .$$

$$N(q,k) = \sum_{i_1 << i_8}^{n} \Delta_{i_1,\ldots,i_8}(q,k) \prod_{h \neq i_1,\ldots,i_8}^{n} D_h + \sum_{i_1 << i_7}^{n} \Delta_{i_1,\ldots,i_7}(q,k) \prod_{h \neq i_1,\ldots,i_7}^{n} D_h + \ldots + \sum_{i_1 << i_2}^{n} \Delta_{i_1,i_2}(q,k) \prod_{h \neq i_1,i_2}^{n} D_h \; , \tag{2.5}$$

In Dim-Reg higher-point higher-dim MI's can appear

- Problem: what is the form of the residues?
- “find the right variables encoding the cut-structure”

The Shape of Residues

m-particle cut the vanishing of m denominators present in that diagram.

- Loop momentum decomposition

$$q + p_i = \sum_{\alpha=1}^{4} x_\alpha \, e_\alpha^{(ijk\cdots)} \,, \qquad k + p_i = \sum_{\alpha=1}^{4} y_\alpha \, e_\alpha^{(ijk\cdots)} \,,$$

m-particle residue: $\Delta_{i_1,\ldots,i_m}$

legs	basis	
	external (p_i)	auxiliary (v_i)
5	4	0
4	3	1
3	2	2
2	1	3
1	0	4

ISP's variables in $\Delta_{i_1,\ldots,i_m}$

- ISP's = Irreducible Scalar Products:
  - spurious: vanishing upon integration
  - non-spourios: non-vanishing upon integration $\Rightarrow$ MI's
- @ 2-Loop
  - $(q \cdot p_i)$ and $(k \cdot p_i)$ can be ISP's ($\neq$ 1-Loop)
  - some ISP's could be chosen to be spurious
  - ISP's from:
    - $*$ direct inspection of the cut-solutions
    - $*$ relations among scalar products *via* Gram's Id'y

Badger, Frellesvig, Zhang

Multi-Loop Scattering Amp's from Multivariate Polynomial Division

Algebraic Geometry

- deals with multivariate polynomials in z $= (z_1, z_2, \ldots)$.
- Ideal $\mathcal{J} \equiv \langle \omega_1(\mathsf{z}) \cdots \omega_s(\mathsf{z}) \rangle$ generated by ω_i
  - $\mathcal{J} = \{ \sum_i h_i(\mathsf{z})\ \omega_i(\mathsf{z}) \}$
  - polynomial coefficients $h_i(\mathsf{z})$
- Multivariate polynomial division of $f(\mathsf{z})$ modulo $\omega_1(\mathsf{z}), \ldots, \omega_s(\mathsf{z})$
  - needs an order, i.e. $z_1 z_2 \overset{?}{>} z_1^2$
  - $\rightsquigarrow f(\mathsf{z}) = \sum_i h_i(\mathsf{z}) \omega_i(\mathsf{z}) + \mathcal{R}(\mathsf{z})$
  - $h_i(\mathsf{z})$ & $\mathcal{R}(\mathsf{z})$ not unique
- Gröbner basis $\{g_1(\mathsf{z}), \ldots, g_r(\mathsf{z})\}$
  - exists (Buchberger's algorithm) & generates $\mathcal{J}$
  - $\rightsquigarrow$ unique $\mathcal{R}(\mathsf{z})$
- Hilbert's Nullstellensatz
  - $V(\mathcal{J}) =$ set of common zeros of $\mathcal{J}$
  - ($f = 0$ in $V(\mathcal{J})$) $\Rightarrow$ ($f^r \in \mathcal{J}$ for some r)
  - Weak Nullstellensatz: ($V(\mathcal{J}) = \varnothing$) $\Leftrightarrow$ ($1 \in \mathcal{J}$)

Multivariate Polynomial Division

Zhang (2012);
Mirabella, Ossola, Peraro, & P.M. (2012)

- **Ideal** $$\mathcal{J}_{i_1\cdots i_n} = \langle D_{i_1}, \cdots, D_{i_n} \rangle \equiv \left\{ \sum_{\kappa=1}^{n} h_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) : h_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

- **Groebner Basis** $$\mathcal{G}_{i_1\cdots i_n} = \{g_1(\mathbf{z}), \ldots, g_m(\mathbf{z})\}$$

  n-ple cut-conditions $$D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$$

Multivariate Polynomial Division

Zhang (2012);
Mirabella, Ossola, Peraro, & P.M. (2012)

Ideal $\mathcal{J}_{i_1\cdots i_n} = \langle D_{i_1}, \cdots, D_{i_n} \rangle \equiv \left\{ \sum_{\kappa=1}^{n} h_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) : h_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$

Groebner Basis $\mathcal{G}_{i_1\cdots i_n} = \{g_1(\mathbf{z}), \ldots, g_m(\mathbf{z})\}$

n-ple cut-conditions $D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$

Polynomial Division $\mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) = \Gamma_{i_1\cdots i_n} + \Delta_{i_1\cdots i_n}(\mathbf{z})$,

Remainder ~ Residue $\Delta_{i_1\cdots i_n}(\mathbf{z})$

Quotients $\Gamma_{i_1\cdots i_n} = \sum_{i=1}^{m} \mathcal{Q}_i(\mathbf{z}) g_i(\mathbf{z})$ belongs to the ideal $\mathcal{J}_{i_1\cdots i_n}$,

$$= \sum_{\kappa=1}^{n} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \; .$$

Multi-Loop Recursive Integrand Reduction

Mirabella, Ossola, Peraro, & P.M. (2012)

remainder = residue

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{\kappa=1}^{k} \mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1} i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} \,.$$

n-denominator integrand

(n-1)-denominator integrand

Reducibility Criterion

Mirabella, Ossola, Peraro, & P.M. (2012)

Proposition 2.1. *The integrand* $\mathcal{I}_{i_1\cdots i_n}$ *is reducible iff the remainder of the division modulo a Gröbner basis vanishes, i.e. iff* $\mathcal{N}_{i_1\cdots i_n} \in \mathcal{J}_{i_1\cdots i_n}$.

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Proposition 2.2. *Any n-particle integrand with $n > 4\ell$ is reducible.*

Proof. In this case, the system is over-constrained, namely the number n of equations is larger than the number 4ℓ of indeterminates. The n propagators cannot vanish simultaneously, i.e.

$$D_{i_1}(\mathbf{z}) = \cdots = D_{i_n}(\mathbf{z}) = 0 \tag{2.7}$$

has no solution. Therefore, according to the *weak Nullstellensatz* theorem

$$1 = \sum_{\kappa=1}^{n} w_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \ \in \mathcal{J}_{i_1\cdots i_n} \ , \tag{2.8}$$

for some $\omega_\kappa \in P[\mathbf{z}]$. Irrespective of the monomial order, a (reduced) Gröbner basis is $\mathcal{G} = \{g_1\} = \{1\}$. Eq. (2.5) becomes

$$\mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) = \mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) \times 1 \ \in \mathcal{J}_{i_1\cdots i_n} \ , \tag{2.9}$$

thus $\mathcal{I}_{i_1\cdots i_n}$ is reducible. □

One-Loop Integrand Reduction

In d-dimensions, the generic n-point one-loop integrand reads $\quad \mathcal{I}_{0\cdots(n-1)} \equiv \dfrac{\mathcal{N}_{0\cdots(n-1)}(q,\mu^2)}{D_0(q,\mu^2)\cdots D_{n-1}(q,\mu^2)}$.

for each $\mathcal{I}_{i_1\cdots i_k}$ we define a basis $\mathcal{E}^{(i_1\cdots i_k)} = \{e_i\}_{i=1,\dots,4}$.

If $k \geq 5$, then $e_i = k_i$, where k_i are four external momenta.
If $k < 5$, then e_i are chosen to fulfill the following relations:

$$e_1^2 = e_2^2 = 0\,, \qquad e_1 \cdot e_2 = 1\,,$$
$$e_3^2 = e_4^2 = \delta_{k4}\,, \qquad e_3 \cdot e_4 = -(1-\delta_{k4})\,.$$

In terms of $\mathcal{E}^{(i_1\cdots i_k)}$, the loop momentum can be decomposed as, $\quad q^\mu = -p^\mu_{i_1} + x_1\, e_1^\mu + x_2\, e_2^\mu + x_3\, e_3^\mu + x_4\, e_4^\mu$.

each numerator $\mathcal{N}_{i_1\cdots i_k}$ can be treated as a rank- k polynomial in $\mathbf{z} \equiv (x_1, x_2, x_3, x_4, \mu^2)$,

$$\mathcal{N}_{i_1\cdots i_k} = \sum_{\vec{j}\in J(k)} \alpha_{\vec{j}}\, z_1^{j_1}\, z_2^{j_2}\, z_3^{j_3}\, z_4^{j_4}\, z_5^{j_5}\,,$$

$J(k) \equiv \{\vec{j} = (j_1,\dots,j_5) : j_1 + j_2 + j_3 + j_4 + 2\,j_5 \leq k\}$.

☑ *Step 1.* Since $n > 5$, the Proposition 2.2 guarantees that $\mathcal{N}_{0\cdots n-1}$ is reducible, and, by iteration, it can be written as a linear combination of 5-point integrands $\mathcal{I}_{i_1\cdots i_5}$.

☑ *Step 1.* Since $n > 5$, the Proposition 2.2 guarantees that $\mathcal{N}_{0\cdots n-1}$ is reducible, and, by iteration, it can be written as a linear combination of 5-point integrands $\mathcal{I}_{i_1\cdots i_5}$.

☑ *Step 2.* The numerator of each $\mathcal{I}_{i_1\cdots i_5}$ is a rank-5 polynomial in $\mathbf{z}$. We define the ideal $\mathcal{J}_{i_1\cdots i_5}$, and compute the Gröbner basis $\mathcal{G}_{i_1\cdots i_5} = (g_1, \ldots, g_5)$, which is found to have a remarkably simple form:

$$g_i(\mathbf{z}) = c_i + z_i \ , (i = 1, \ldots, 5) \ .$$

[keep it in mind!]

The division of $\mathcal{N}_{i_1\cdots i_5}$ modulo $\mathcal{G}_{i_1\cdots i_5}$ gives a *constant* remainder,

$$\Delta_{i_1\cdots i_5} = c_0 \ .$$

$$\Gamma_{i_1\cdots i_5} = \sum_{\kappa=1}^{5} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_5}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \ ,$$

where $\mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_5}$ are the numerators of the 4-point integrands, $\mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_5}$, obtained by removing the i_κ-th denominator.

☑ *Step 3.* For each $\mathcal{I}_{i_1\cdots i_4}$, the numerator $\mathcal{N}_{i_1\cdots i_4}$ is a rank-4 polynomial in $\mathbf{z}$. The Gröbner basis $\mathcal{G}_{i_1\cdots i_4}$ of the ideal $\mathcal{J}_{i_1\cdots i_4}$ contains four elements. Dividing $\mathcal{N}_{i_1\cdots i_4}$ modulo $\mathcal{G}_{i_1\cdots i_4}$, we obtain

$$\Delta_{i_1\cdots i_4} = c_0 + c_1 x_4 + \mu^2(c_2 + c_3 x_4 + \mu^2 c_4)\,.$$

$$\Gamma_{i_1\cdots i_4} = \sum_{\kappa=1}^{4} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_4}(\mathbf{z}) D_{i_\kappa}(\mathbf{z})\ ,$$

contains the numerators of 3-point integrands $\mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_4}$.

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$$\Delta_{i_1\cdots i_4} = c_0 + c_1 x_4 + \mu^2(c_2 + c_3 x_4 + \mu^2 c_4)\,.$$

$$\Gamma_{i_1\cdots i_4} = \sum_{\kappa=1}^{4} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_4}(\mathbf{z}) D_{i_\kappa}(\mathbf{z})\ ,$$

contains the numerators of 3-point integrands $\mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_4}$.

☑ *Step 4.* The Gröbner basis $\mathcal{G}_{i_1 i_2 i_3}$ is formed by three elements, and is used to divide $\mathcal{N}_{i_1 i_2 i_3}$. The remainder $\Delta_{i_1 i_2 i_3}$ is polynomial in μ^2 and in the third and fourth components of q in the basis $\mathcal{E}^{(i_1 i_2 i_3)}$,

$$\Delta_{i_1 i_2 i_3} = c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2(c_7 + c_8 x_3 + c_9 x_4)\ .$$

The term $\Gamma_{i_1 i_2 i_3}$ generates the rank-2 numerators of the 2-point integrands $\mathcal{I}_{i_1 i_2}$, $\mathcal{I}_{i_1 i_3}$, and $\mathcal{I}_{i_2 i_3}$.

☑ *Step 5.* The remainder of the division of $\mathcal{N}_{i_1 i_2}$ by the two elements of $\mathcal{G}_{i_1 i_2}$ is:

$$\Delta_{i_1 i_2} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \ .$$

It is polynomial in μ^2 and in the last three components of q in the basis $\mathcal{E}^{(i_1 i_2)}$. The reducible term of the division, $\Gamma_{i_1 i_2}$, generates the rank-1 integrands, $\mathcal{I}_{i_1}$, and $\mathcal{I}_{i_2}$.

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$$\Delta_{i_1 i_2} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \ .$$

It is polynomial in μ^2 and in the last three components of q in the basis $\mathcal{E}^{(i_1 i_2)}$. The reducible term of the division, $\Gamma_{i_1 i_2}$, generates the rank-1 integrands, $\mathcal{I}_{i_1}$, and $\mathcal{I}_{i_2}$.

☑ *Step 6.* The numerator of the 1-point integrands is linear in the components of the loop momentum in the basis $\mathcal{E}^{(i_1)}$,

$$\mathcal{N}_{i_1} = \beta_0 + \sum_{j=1}^{4} \beta_j \, x_j \ .$$

The only element of the Gröbner basis $\mathcal{G}_{i_1}$ is D_{i_1}, which is quadratic in $\mathbf{z}$. Therefore the division modulo $\mathcal{G}_{i_1}$, leads to a vanishing quotient, hence

$$\mathcal{N}_{i_1} = \Delta_{i_1} \ .$$

☑ *Step 7.* Collecting all the remainders computed in the previous steps, we obtain the complete decomposition of $\mathcal{I}_{0\cdots n-1}$ in terms of its multi-pole structure

$$\mathcal{I}_{0\cdots n-1} = \sum_{k=1}^{5} \left(\sum_{1=i_1<\ldots<i_k}^{n-1} \frac{\Delta_{i_1\cdots i_k}}{D_{i_1}\cdots D_{i_k}} \right) .$$

which reproduces the well-known one-loop d-dimensional integrand decomposition formula

Ossola, Papadopoulos, Pittau
Ellis, Giele, Kunszt, Melnikov

```
-------  SPINORS @ MATHEMATICA (S@M)  -------


        Version:  S@M 1.0 (3-APR-2007)

 Authors:
    Daniel Maitre (SLAC),
    Pierpaolo Mastrolia (University of Zurich)

A list of all functions provided by the package
is stored in the variable
    $SpinorsFunctions
```

DeclareSpinor[p[1], p[2], p[3], p[4], p[5]]

```
{p[1], p[2], p[3], p[4], p[5]} added to the list of spinors
```

GenMomenta[{p[1], p[2], p[3], p[4], p[5]}]

```
Momenta for the spinors p[1], p[2], p[3], p[4], p[5] generated.
```

DeclareLVector[q, qtemp]

```
{q, qtemp} added to the list of Lorentz vectors
```

DeclareLVector[e[1], e[2], e[3], e[4]]

```
{e[1], e[2], e[3], e[4]} added to the list of Lorentz vectors
```

GroebnerBasis[{$poly_1$, $poly_2$, …}, {x_1, x_2, …}] gives a list of polynomials that form a Gröbner basis for the set of polynomials $poly_i$.

PolynomialReduce[$poly$, {$poly_1$, $poly_2$, …}, {x_1, x_2, …}] yields a list representing a reduction of $poly$ in terms of the $poly_i$. The list has the form {{a_1, a_2, …}, b}, where b is minimal and $a_1\, poly_1 + a_2\, poly_2 + \ldots + b$ is exactly $poly$. ≫

5ple-cut

q-decomposition

```
qdeco = Sum[x[i] * p[i], {i, 1, 4}]
```

```
p[1] x[1] + p[2] x[2] + p[3] x[3] + p[4] x[4]
```

Explicit expression of denominators in terms of x[i]

```
For[i = 0, i ≤ 4, i++,
  Dnr[i] = Den[i];
  Print["D_", i, " = ", Dnr[i]];
 ];
```

```
D_0 = -μ2 + MP[q, q]

D_1 = -μ2 + MP[q, q] + 2 MP[q, p[1]]

D_2 = -μ2 + MP[q, q] + 2 MP[q, p[1]] + 2 MP[q, p[2]] + 2 MP[p[1], p[2]]

D_3 = -μ2 + MP[q, q] + 2 MP[q, p[1]] + 2 MP[q, p[2]] +
  2 MP[q, p[3]] + 2 MP[p[1], p[2]] + 2 MP[p[1], p[3]] + 2 MP[p[2], p[3]]

D_4 = -μ2 + MP[q, q] - 2 MP[q, p[5]]
```

```
D_0 = -1. μ2 + 3.89861 x[1.] x[2.] + 10.8202 x[1.] x[3.] +
  6.38478 x[2.] x[3.] - 2.95514 x[1.] x[4.] - 3.47983 x[2.] x[4.] - 14.6687 x[3.] x[4.]

D_1 = -1. μ2 + 3.89861 x[2.] + 3.89861 x[1.] x[2.] +
  10.8202 x[3.] + 10.8202 x[1.] x[3.] + 6.38478 x[2.] x[3.] - 2.95514 x[4.] -
  2.95514 x[1.] x[4.] - 3.47983 x[2.] x[4.] - 14.6687 x[3.] x[4.]

D_2 = 3.89861 - 1. μ2 + 3.89861 x[1.] + 3.89861 x[2.] + 3.89861 x[1.] x[2.] +
 17.205 x[3.] + 10.8202 x[1.] x[3.] + 6.38478 x[2.] x[3.] - 6.43497 x[4.] -
 2.95514 x[1.] x[4.] - 3.47983 x[2.] x[4.] - 14.6687 x[3.] x[4.]

D_3 = 21.1036 - 1. μ2 + 14.7189 x[1.] + 10.2834 x[2.] + 3.89861 x[1.] x[2.] +
 17.205 x[3.] + 10.8202 x[1.] x[3.] + 6.38478 x[2.] x[3.] - 21.1036 x[4.] -
 2.95514 x[1.] x[4.] - 3.47983 x[2.] x[4.] - 14.6687 x[3.] x[4.]

D_4 = -1. μ2 + 11.7637 x[1.] + 6.80356 x[2.] + 3.89861 x[1.] x[2.] +
  2.53636 x[3.] + 10.8202 x[1.] x[3.] + 6.38478 x[2.] x[3.] - 21.1036 x[4.] -
  2.95514 x[1.] x[4.] - 3.47983 x[2.] x[4.] - 14.6687 x[3.] x[4.]
```

```
Vars = Union[Table[x[i], {i, 1, 4}]] // N;
Vars = Append[Vars, μ2]
```

```
{x[1.], x[2.], x[3.], x[4.], μ2}
```

```
GB = GroebnerBasis[Cut[0, 1, 2, 3, 4], Vars];
```

```
For[i = 1, i ≤ Length[GB], i++,
 Print[i, " : ", GB[[i]]];
]
```

```
1 : 31.9034 + 1. μ2

2 : 3.44658 + 1. x[4.]

3 : -0.196787 + 1. x[3.]

4 : 3.15867 + 1. x[2.]

5 : 4.39864 + 1. x[1.]
```

```
Num5 = (
  c[0] +
   Sum[c[i1] * Vars[[i1]], {i1, 1, Length[Vars]}] +
   Sum[c[i1, i2] * Vars[[i1]] * Vars[[i2]],
    {i1, 1, Length[Vars]}, {i2, i1, Length[Vars]}] +
   Sum[c[i1, i2, i3] * Vars[[i1]] * Vars[[i2]] * Vars[[i3]],
    {i1, 1, Length[Vars]}, {i2, i1, Length[Vars]}, {i3, i2, Length[Vars]}] +
   Sum[c[i1, i2, i3, i4] * Vars[[i1]] * Vars[[i2]] * Vars[[i3]] * Vars[[i4]],
    {i1, 1, Length[Vars]}, {i2, i1, Length[Vars]},
    {i3, i2, Length[Vars]}, {i4, i3, Length[Vars]}] +
   Sum[c[i1, i2, i3, i4, i5] * Vars[[i1]] * Vars[[i2]] * Vars[[i3]] *
     Vars[[i4]] * Vars[[i5]], {i1, 1, Length[Vars]}, {i2, i1, Length[Vars]},
    {i3, i2, Length[Vars]}, {i4, i3, Length[Vars]}, {i5, i4, Length[Vars]}]
 )
```

$$c[0] + \mu 2\, c[5] + \mu 2^2\, c[5,5] + \mu 2^3\, c[5,5,5] + \mu 2^4\, c[5,5,5,5] + \mu 2^5\, c[5,5,5,5,5] + c[1]\, x[1.] + \mu 2\, c[1,5]\, x[1.] + \mu 2^2\, c[1,5,5]\, x[1.] + \mu 2^3\, c[1,5,5,5]\, x[1.] + \mu 2^4\, c[1,5,5,5,5]\, x[1.] + c[1,1]\, x[1.]^2 + \mu 2\, c[1,1,5]\, x[1.]^2 + \mu 2^2\, c[1,1,5,5]\, x[1.]^2 + \mu 2^3\, c[1,1,5,5,5]\, x[1.]^2 + c[1,1,1]\, x[1.]^3 + \mu 2\, c[1,1,1,5]\, x[1.]^3 + \mu 2^2\, c[1,1,1,5,5]\, x[1.]^3 + c[1,1,1,1]\, x[1.]^4 + \mu 2\, c[1,1,1,1,5]\, x[1.]^4 + c[1,1,1,1,1]\, x[1.]^5 + c[2]\, x[2.] + \mu 2\, c[2,5]\, x[2.] + \mu 2^2\, c[2,5,5]\, x[2.] + \mu 2^3\, c[2,5,5,5]\, x[2.] + \mu 2^4\, c[2,5,5,5,5]\, x[2.] + c[1,2]\, x[1.]\, x[2.] + \ldots$$

```
Length[%]
```

```
252
```

- **5-point residue**

```
resto = PolynomialReduce[Num5, GB, Vars][[2]];

Length[MonomialList[resto, Vars]]

1
```

4ple-cut

- **q-decomposition**

```
qdeco = Sum[x[i] * e[i], {i, 1, 2}] + x[3] * v + x[4] * vperp
```

$$e[1]\,x[1] + e[2]\,x[2] + \frac{1}{2}\,(e[3] + e[4])\,x[3] + \frac{1}{2}\,(e[3] - e[4])\,x[4]$$

Note that (e[3]-e[4])/2 = vperp, because it is ortogonal to e[1], e[2] and v=(e[3]+e[4])/2

- **Explicit expression of denominators in terms of x[i]**

$$D\_0 = -1.\,\mu 2 - 3.92177\,x[1.]\,x[2.] + 0.980442\,x[3.]^2 - 0.980442\,x[4.]^2$$

$$D\_1 = -1.\,\mu 2 - 3.92177\,x[2.] - 3.92177\,x[1.]\,x[2.] + 0.980442\,x[3.]^2 - 0.980442\,x[4.]^2$$

$$D\_2 = 12.4329 - 1.\,\mu 2 + 12.4329\,x[1.] - 7.85602\,x[2.] - 3.92177\,x[1.]\,x[2.] + (0. + 4.26815\,i)\,x[3.] + 0.980442\,x[3.]^2 + 5.54049\,x[4.] - 0.980442\,x[4.]^2$$

$$D\_3 = 4.43686 - 1.\,\mu 2 + 11.6475\,x[1.] - 4.43686\,x[2.] - 3.92177\,x[1.]\,x[2.] + (0. + 2.66221\,i)\,x[3.] + 0.980442\,x[3.]^2 + 5.21447\,x[4.] - 0.980442\,x[4.]^2$$

$$D\_4 = -1.\,\mu 2 + 3.92177\,x[1.] - 3.92177\,x[1.]\,x[2.] + 0.980442\,x[3.]^2 - 0.980442\,x[4.]^2$$

```
GB = GroebnerBasis[Cut[0, 1, 2, 3], Vars];
```

1 : $29.1065 + 1.01962\,\mu 2 - 0.193505\,x[4.] + 1.\,x[4.]^2$

2 : $(0. - 5.39591\,i) + 1.\,x[3.] + (0. + 0.0179364\,i)\,x[4.]$

3 : $1.\,x[2.]$

4 : $-0.852388 + 1.\,x[1.] + 0.451789\,x[4.]$

```
Num4
```

$$c[0] + \mu 2\,c[5] + \mu 2^2\,c[5, 5] + c[1]\,x[1.] + \mu 2\,c[1, 5]\,x[1.] + c[1, 1]\,x[1.]^2 + \mu 2\,c[1, 1, 5]\,x[1.]^2 + c[1, 1, 1]\,x[1.]^3 + c[1, 1, 1, 1]\,x[1.]^4 + c[2]\,x[2.] + \mu 2\,c[2, 5]\,x[2.] + c[1, 2]\,x[1.]\,x[2.] + \mu 2\,c[1, 2, 5]\,x[1.]\,x[2.] + c[1, 1, 2]\,x[1.]^2\,x[2.] + c[1, 1, 1, 2]\,x[1.]^3\,x[2.] + c[2, 2]\,x[2.]^2 + \mu 2\,c[2, 2, 5]\,x[2.]^2 + c[1, 2, 2]\,x[1.]\,x[2.]^2 + c[1, 1, 2, 2]\,x[1.]^2\,x[2.]^2 + c[2, 2, 2]\,x[2.]^3 + c[1, 2, 2, 2]\,x[1.]\,x[2.]^3 + c[2, 2, 2, 2]\,x[2.]^4 + c[3]\,x[3.] + \mu 2\,c[3, 5]\,x[3.] + c[1, 3]\,x[1.]\,x[3.] + \mu 2\,c[1, 3, 5]\,x[1.]\,x[3.] + c[1, 1, 3]\,x[1.]^2\,x[3.] + c[1, 1, 1, 3]\,x[1.]^3\,x[3.] + \ldots$$

```
Length[%]
```

86

- **4-point residue**

```
resto = PolynomialReduce[Num4, GB, Vars, MonomialOrder → Lexicographic][[2]];

Length[MonomialList[resto, Vars]]

5
```

3ple-cut

- **q-decomposition**

```
qdeco = Sum[x[i] * e[i], {i, 1, 4}]

e[1] x[1] + e[2] x[2] + e[3] x[3] + e[4] x[4]
```

- **Explicit expression of denominators in terms of x[i]**

```
D_0 = -1. μ2 - 11.7637 x[1.] x[2.] + 11.7637 x[3.] x[4.]

D_1 = -1. μ2 - 11.7637 x[2.] - 11.7637 x[1.] x[2.] + 11.7637 x[3.] x[4.]

D_2 = 3.89861 - 1. μ2 + 3.89861 x[1.] - 18.5673 x[2.] - 11.7637 x[1.] x[2.] -
  (0.432545 - 5.13199 i) x[3.] + (0.432545 + 5.13199 i) x[4.] + 11.7637 x[3.] x[4.]

D_3 = 21.1036 - 1. μ2 + 14.7189 x[1.] - 21.1036 x[2.] - 11.7637 x[1.] x[2.] +
  (4.65563 + 6.37882 i) x[3.] - (4.65563 - 6.37882 i) x[4.] + 11.7637 x[3.] x[4.]

D_4 = -1. μ2 + 11.7637 x[1.] - 11.7637 x[1.] x[2.] + 11.7637 x[3.] x[4.]
```

```
GB = GroebnerBasis[Cut[0, 1, 4], Vars];
```

```
For[i = 1, i ≤ Length[GB], i++,
 Print[i, " : ", GB[[i]]];
]

1 : -0.0850072 μ2 + 1. x[3.] x[4.]

2 : 1. x[2.]

3 : 1. x[1.]
```

```
Num3

c[0] + μ2 c[5] + c[1] x[1.] + μ2 c[1, 5] x[1.] + c[1, 1] x[1.]^2 + c[1, 1, 1] x[1.]^3 +
  c[2] x[2.] + μ2 c[2, 5] x[2.] + c[1, 2] x[1.] x[2.] + c[1, 1, 2] x[1.]^2 x[2.] +
  c[2, 2] x[2.]^2 + c[1, 2, 2] x[1.] x[2.]^2 + c[2, 2, 2] x[2.]^3 + c[3] x[3.] +
  μ2 c[3, 5] x[3.] + c[1, 3] x[1.] x[3.] + c[1, 1, 3] x[1.]^2 x[3.] + c[2, 3] x[2.] x[3.] +
  c[1, 2, 3] x[1.] x[2.] x[3.] + c[2, 2, 3] x[2.]^2 x[3.] + c[3, 3] x[3.]^2 +
  c[1, 3, 3] x[1.] x[3.]^2 + c[2, 3, 3] x[2.] x[3.]^2 + c[3, 3, 3] x[3.]^3 +
  c[4] x[4.] + μ2 c[4, 5] x[4.] + c[1, 4] x[1.] x[4.] + c[1, 1, 4] x[1.]^2 x[4.] +
  c[2, 4] x[2.] x[4.] + c[1, 2, 4] x[1.] x[2.] x[4.] + c[2, 2, 4] x[2.]^2 x[4.] +
  c[3, 4] x[3.] x[4.] + c[1, 3, 4] x[1.] x[3.] x[4.] + c[2, 3, 4] x[2.] x[3.] x[4.] +
  c[3, 3, 4] x[3.]^2 x[4.] + c[4, 4] x[4.]^2 + c[1, 4, 4] x[1.] x[4.]^2 +
  c[2, 4, 4] x[2.] x[4.]^2 + c[3, 4, 4] x[3.] x[4.]^2 + c[4, 4, 4] x[4.]^3

Length[%]

40
```

- **3-point residue**

```
resto = PolynomialReduce[Num3, GB, Vars, MonomialOrder → Lexicographic][[2]];

Length[MonomialList[resto, Vars]]

10

Nresto

c[0.] + 1. μ2 c[5.] + 0.0850072 μ2 c[3., 4.] + c[3.] x[3.] +
 1. μ2 c[3., 5.] x[3.] + 0.0850072 μ2 c[3., 3., 4.] x[3.] + c[3., 3.] x[3.]^2 +
 c[3., 3., 3.] x[3.]^3 + c[4.] x[4.] + 1. μ2 c[4., 5.] x[4.] +
 0.0850072 μ2 c[3., 4., 4.] x[4.] + c[4., 4.] x[4.]^2 + c[4., 4., 4.] x[4.]^3
```

2ple-cut

- **q-decomposition**

```
qdeco = Sum[x[i] * e[i], {i, 1, 4}]
```

```
e[1] x[1] + e[2] x[2] + e[3] x[3] + e[4] x[4]
```

- **Explicit expression of denominators in terms of x[i]**

```
D_0 = -1. μ2 - 11.7637 x[1.] x[2.] + 11.7637 x[3.] x[4.]

D_1 = -1. μ2 - 11.7637 x[2.] - 11.7637 x[1.] x[2.] + 11.7637 x[3.] x[4.]

D_2 = 3.89861 - 1. μ2 + 3.89861 x[1.] - 18.5673 x[2.] - 11.7637 x[1.] x[2.] -
  (0.432545 - 5.13199 i) x[3.] + (0.432545 + 5.13199 i) x[4.] + 11.7637 x[3.] x[4.]

D_3 = 21.1036 - 1. μ2 + 14.7189 x[1.] - 21.1036 x[2.] - 11.7637 x[1.] x[2.] +
  (4.65563 + 6.37882 i) x[3.] - (4.65563 - 6.37882 i) x[4.] + 11.7637 x[3.] x[4.]

D_4 = -1. μ2 + 11.7637 x[1.] - 11.7637 x[1.] x[2.] + 11.7637 x[3.] x[4.]
```

```
GB = GroebnerBasis[Cut[0, 4], Vars];
```

```
1 : -0.0850072 μ2 + 1. x[3.] x[4.]

2 : 1. x[1.]
```

```
Num2
```

```
c[0] + μ2 c[5] + c[1] x[1.] + c[1, 1] x[1.]^2 + c[2] x[2.] + c[1, 2] x[1.] x[2.] + c[2, 2] x[2.]^2 +
  c[3] x[3.] + c[1, 3] x[1.] x[3.] + c[2, 3] x[2.] x[3.] + c[3, 3] x[3.]^2 + c[4] x[4.] +
  c[1, 4] x[1.] x[4.] + c[2, 4] x[2.] x[4.] + c[3, 4] x[3.] x[4.] + c[4, 4] x[4.]^2
```

```
Length[MonomialList[Num2, Vars]]
```

```
16
```

- **2-point residue**

```
resto = PolynomialReduce[Num2, GB, Vars, MonomialOrder → Lexicographic][[2]];

Length[MonomialList[resto, Vars]]
```

```
10
```

```
Nresto
```

```
c[0.] + 1. μ2 c[5.] + 0.0850072 μ2 c[3., 4.] +
  c[2.] x[2.] + c[2., 2.] x[2.]^2 + c[3.] x[3.] + c[2., 3.] x[2.] x[3.] +
  c[3., 3.] x[3.]^2 + c[4.] x[4.] + c[2., 4.] x[2.] x[4.] + c[4., 4.] x[4.]^2
```

1ple-cut

▪ q-decomposition

```
qdeco = Sum[x[i] * e[i], {i, 1, 4}]
```

```
e[1] x[1] + e[2] x[2] + e[3] x[3] + e[4] x[4]
```

▪ Explicit expression of denominators in terms of x[i]

```
D_0 = -1. μ2 - 11.7637 x[1.] x[2.] + 11.7637 x[3.] x[4.]

D_1 = -1. μ2 - 11.7637 x[2.] - 11.7637 x[1.] x[2.] + 11.7637 x[3.] x[4.]

D_2 = 3.89861 - 1. μ2 + 3.89861 x[1.] - 18.5673 x[2.] - 11.7637 x[1.] x[2.] -
  (0.432545 - 5.13199 i) x[3.] + (0.432545 + 5.13199 i) x[4.] + 11.7637 x[3.] x[4.]

D_3 = 21.1036 - 1. μ2 + 14.7189 x[1.] - 21.1036 x[2.] - 11.7637 x[1.] x[2.] +
  (4.65563 + 6.37882 i) x[3.] - (4.65563 - 6.37882 i) x[4.] + 11.7637 x[3.] x[4.]

D_4 = -1. μ2 + 11.7637 x[1.] - 11.7637 x[1.] x[2.] + 11.7637 x[3.] x[4.]
```

```
GB = GroebnerBasis[Cut[0], Vars];
```

```
For[i = 1, i ≤ Length[GB], i++,
  Print[i, " : ", GB[[i]]];
]
```

```
1 : 0.0850072 μ2 + 1. x[1.] x[2.] - 1. x[3.] x[4.]
```

```
Num1
```

```
c[0] + c[1] x[1.] + c[2] x[2.] + c[3] x[3.] + c[4] x[4.]
```

```
Length[MonomialList[Num1, Vars]]
```

```
5
```

▪ 1-point residue

```
resto = PolynomialReduce[Num1, GB, Vars, MonomialOrder → Lexicographic][[2]];
```

```
Length[MonomialList[resto, Vars]]
```

```
5
```

```
Nresto
```

```
c[0.] + c[1.] x[1.] + c[2.] x[2.] + c[3.] x[3.] + c[4.] x[4.]
```

```
quoz = PolynomialReduce[Num1, GB, Vars, MonomialOrder → Lexicographic][[1]]
```

```
{0.}
```

- What can we do within this new framework?

The Maximum-Cut Theorem

Mirabella, Ossola, Peraro, & P.M. (2012)

At ℓ loops, in four dimensions, we define a *maximum-cut* as a (4ℓ)-ple cut

$$D_{i_1} = D_{i_2} = \cdots = D_{i_{4\ell}} = 0 \ ,$$

which constrains completely the components of the loop momenta. In four dimensions this implies the presence of four constraints for each loop momenta.

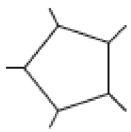
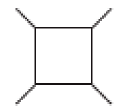
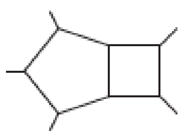
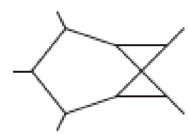
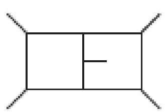
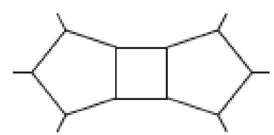
We assume that:

in non-exceptional phase-space points, a maximum-cut has a finite number n_s of solutions, each with multiplicity one.

Under this assumption we have the following

Theorem 4.1 (Maximum cut). *The residue at the maximum-cut is a polynomial parametrised by n_s coefficients, which admits a univariate representation of degree $(n_s - 1)$.*

Examples of Maximum-Cuts

diagram	Δ	n_s	diagram	Δ	n_s
	c_0	1		$c_0 + c_1 z$	2
	$\sum_{i=0}^{3} c_i z^i$	4		$\sum_{i=0}^{3} c_i z^{\,i}$	4
	$\sum_{i=0}^{7} c_i z^{\,i}$	8		$\sum_{i=0}^{7} c_i z^{\,i}$	8

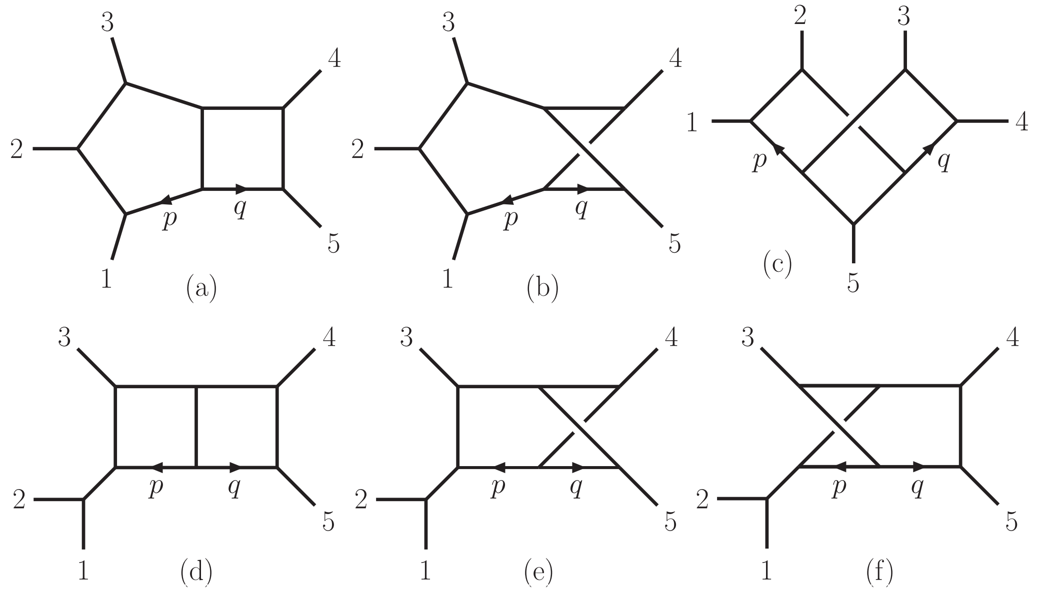
2-loop 5-point amplitudes in N=4 SYM

Bern, Czakon, Kosower, Roiban, Smirnov

Arkani-Hamed, Bourjaily, Cachazo, Caron-Houot, Trnka

Drummond, Henn, Trnka

Carrasco, Johansson



Integrand

$$\mathcal{I}_{1\cdots 8} \equiv \frac{\mathcal{N}_{1\cdots 8}(q,k)}{D_1(q,k)\cdots D_8(q,k)} \, ,$$

Momentum basis

$$q^\mu = \sum_{i=1}^{4} y_i \, \tau_i^\mu \, , \qquad k^\mu = \sum_{i=1}^{4} x_i \, e_i^\mu \, .$$

Generic Numerator

$$\mathcal{N}(x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) = \sum_{\vec{j}\in J(k)} \alpha_{\vec{j}} \, x_1^{j_1} \, x_2^{j_2} \, x_3^{j_3} \, x_4^{j_4} \, y_1^{j_5} \, y_2^{j_6} \, y_3^{j_7} \, y_4^{j_8} \, ,$$

with $J(k)$ being the set of values for the exponents compatible with the renormalizability

Polynomial Division

$$\mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) = \sum_{\kappa=1}^{n} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) + \Delta_{i_1\cdots i_n}(\mathbf{z}) \, .$$

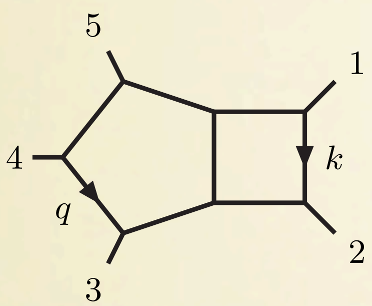
2-Loop Integrand Decomposition Formula (4D)

$$\mathcal{I}_n = \sum_{i_1<<i_8=1}^{n} \frac{\Delta_{i_1\cdots i_8}}{D_{i_1}\cdots D_{i_8}} + \sum_{i_1<<i_7=1}^{n} \frac{\Delta_{i_1\cdots i_7}}{D_{i_1}\cdots D_{i_7}} + \cdots + \sum_{i_1<i_2=1}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{i=1}^{n} \frac{\Delta_i}{D_i} + \mathcal{Q}_\varnothing$$

The PentaBox diagram in N=4 SYM

Ossola & P.M. (2011)

Mirabella, Ossola, Peraro, & P.M. (in progress)

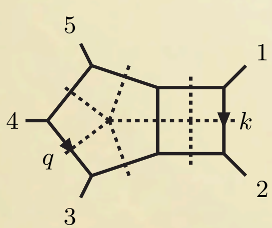


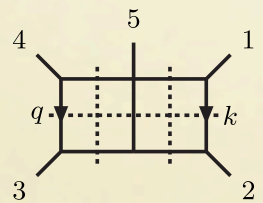
$$
\begin{aligned}
D_1 &= k^2 \\
D_2 &= (k+p_2)^2 \\
D_3 &= (k-p_1)^2 \\
D_4 &= q^2 \\
D_5 &= (q+p_3)^2 \\
D_6 &= (q-p_4)^2 \\
D_7 &= (q-p_4-p_5)^2 \\
D_8 &= (q+k+p_2+p_3)^2 \; .
\end{aligned}
$$

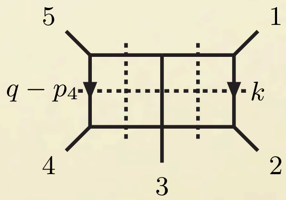
$$N(q,k) = 2\, q \cdot v + \alpha$$ Carrasco & Johansson (2011)

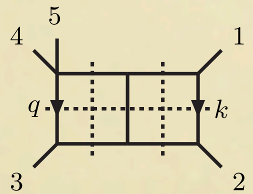
$$v^\mu = \frac{1}{4}\Big(\gamma_{12}(p_1^\mu - p_2^\mu) + \gamma_{23}(p_2^\mu - p_3^\mu) + 2\,\gamma_{45}(p_4^\mu - p_5^\mu) + \gamma_{13}(p_1^\mu - p_3^\mu)\Big)$$

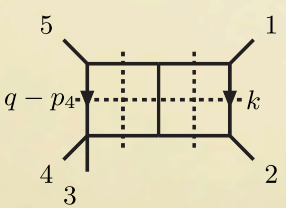
$$\alpha = \frac{1}{4}\Big(2\,\gamma_{12}(s_{45} - s_{12}) + \gamma_{23}(s_{45} + 3 s_{12} - s_{13}) + 2\,\gamma_{45}(s_{14} - s_{15}) + \gamma_{13}(s_{12} + s_{45} - s_{13})\Big)$$



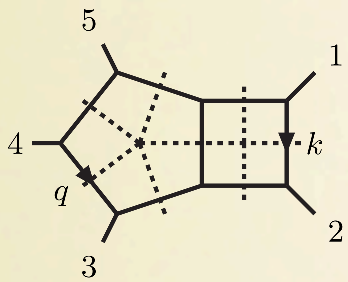








5-POINT 8FOLD-CUT $D_1 = \ldots = D_8 = 0$



$$\Delta_{12345678}(q,k) = \mathrm{Res}_{12345678}\left\{\mathcal{N}_{1\cdots 8}(q,k)\right\} .$$

$$q^\mu = \sum_{i=1}^{4} y_i \, \tau_i^\mu \, , \qquad k^\mu = \sum_{i=1}^{4} x_i \, e_i^\mu \, . \qquad e_1 = p_1, \qquad e_2 = p_2, \qquad \tau_1 = p_3, \qquad \tau_2 = p_4.$$

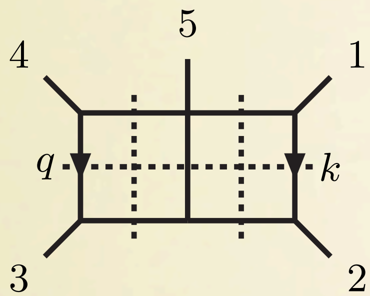
$$\Delta_{12345678}(q,k) = c_{12345678,0} + c_{12345678,1}\, y_4 + c_{12345678,2}\, x_3 + c_{12345678,3}\, x_4 \ .$$

[Maximum Cut Thm]

generic residue

5-point 7fold-cut

$$D_1 = \ldots = D_6 = D_8 = 0$$



$$\Delta_{1234568}(q,k) = \mathrm{Res}_{1234568}\left\{\frac{N(q,k) - \Delta_{12345678}(q,k)}{D_7}\right\}.$$

$$q^\mu = \sum_{i=1}^{4} y_i\, \tau_i^\mu\ , \qquad k^\mu = \sum_{i=1}^{4} x_i\, e_i^\mu\ .$$

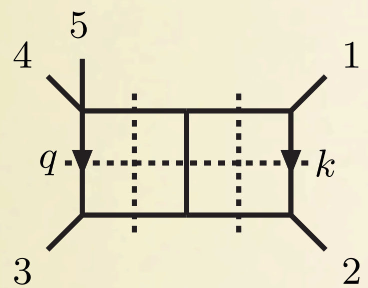
$$e_1 = p_1, \qquad e_2 = p_2, \qquad \tau_1 = p_3, \qquad \tau_2 = p_4.$$

$$\begin{aligned}
\Delta_{1234568} =\ & c_0 + c_1\, x_3 + c_2\, x_3^2 + c_3\, x_3^3 + c_4\, x_3^4 + c_5\, x_4 + c_6\, x_4^2 + c_7\, x_4^3 + c_8\, x_4^4 \\
& + c_9\, y_3 + c_{10}\, x_4\, y_3 + c_{11}\, y_3^2 + c_{12}\, x_4 y_3^2 + c_{13}\, y_3^3 + c_{14}\, x_4 y_3^3 + c_{15}\, y_3^4 \\
& + c_{16}\, x_4 y_3^4 + c_{17}\, y_4 + c_{18}\, x_3 y_4 + c_{19}\, x_3^2 y_4 + c_{20}\, x_3^3 y_4 + c_{21}\, x_3^4 y_4 + c_{22}\, x_4 y_4 \\
& + c_{23}\, x_4^2 y_4 + c_{24}\, x_4^3 y_4 + c_{25}\, x_4^4 y_4 + c_{26}\, y_4^2 + c_{27}\, x_4 y_4^2 + c_{28}\, y_4^3 + c_{29}\, x_4 y_4^3 \\
& + c_{30}\, y_4^4 + c_{31}\, x_4 y_4^4\ .
\end{aligned} \qquad (3.18)$$

generic residue

4-POINT 7FOLD-CUT

$D_1 = \ldots = D_5 = D_7 = D_8 = 0.$



$$\Delta_{1234578}(q,k) = \mathrm{Res}_{1234578}\left\{\frac{N(q,k) - \Delta_{12345678}(q,k)}{D_6}\right\},$$

$$e_1^\mu = p_1^\mu\,, \qquad e_2^\mu = p_2^\mu\,,$$
$$\tau_1^\mu = p_3^\mu\,, \qquad \tau_2^\mu = P_{45}^\mu - \frac{s_{45}}{2P_{45}\cdot\tau_1}\tau_1^\mu\,.$$

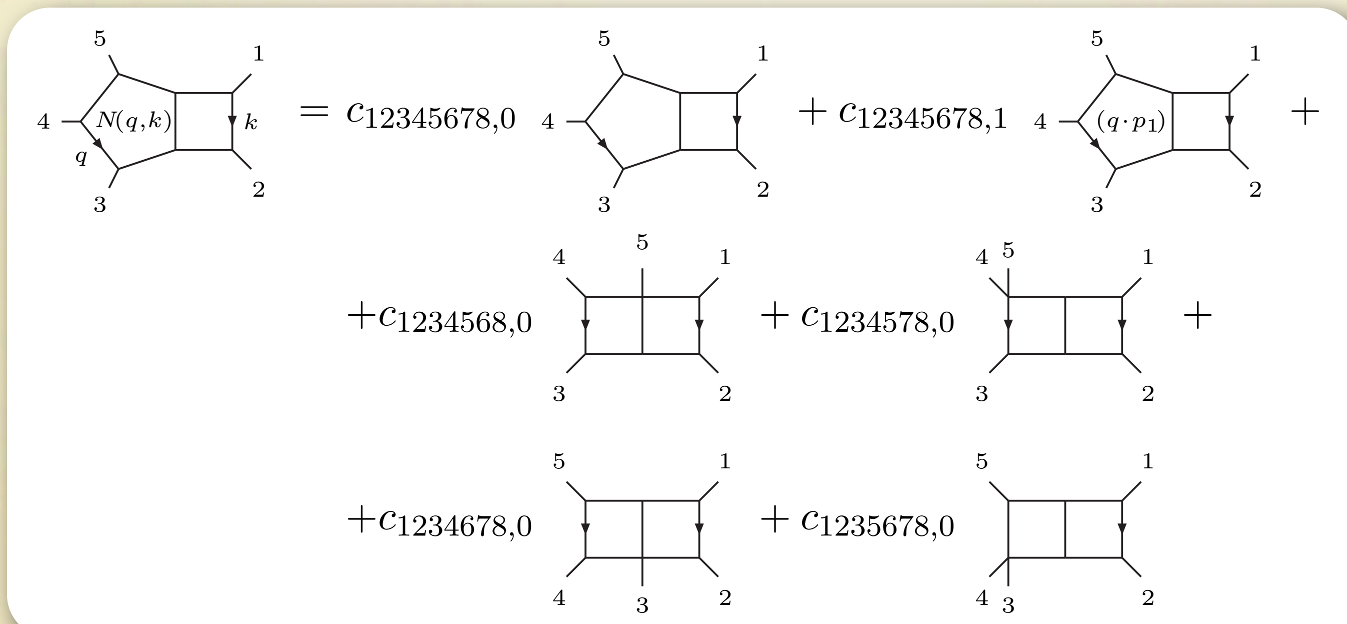
parametrized using thirty-two monomials

$$\{1, x_3, x_3^2, x_3^3, x_3^4, x_4, x_4^2, x_4^3, x_4^4, y_3, x_4y_3, y_3^2, x_4y_3^2, y_3^3, x_4y_3^3, y_3^4, x_4y_3^4, y_4, x_3y_4,$$
$$x_3^2y_4, x_3^3y_4, x_3^4y_4, x_4y_4, x_4^2y_4, x_4^3y_4, x_4^4y_4, y_4^2, x_4y_4^2, y_4^3, x_4y_4^3, y_4^4, x_4y_4^4\}\,.$$

generic residue

PentaBox Integrand Decomposition

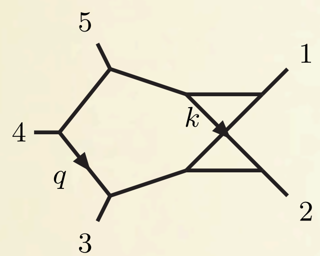
$$
\begin{aligned}
N(q,k) &= \Delta_{12345678}(q,k) + \\
&\quad +\Delta_{1234568}(q,k)D_7 + \Delta_{1234578}(q,k)D_6 + \\
&\quad +\Delta_{1234678}(q,k)D_5 + \Delta_{1235678}(q,k)D_4 = \\
&= c_{12345678,0} + c_{12345678,1}\ (q\cdot p_1) + \\
&\quad +c_{1234568,0}D_7 + c_{1234578,0}D_6 + \\
&\quad +c_{1234678,0}D_5 + c_{1235678,0}D_4\ ,
\end{aligned}
$$



Global (N=N)-test: OK

PentaCross Integrand Decomposition

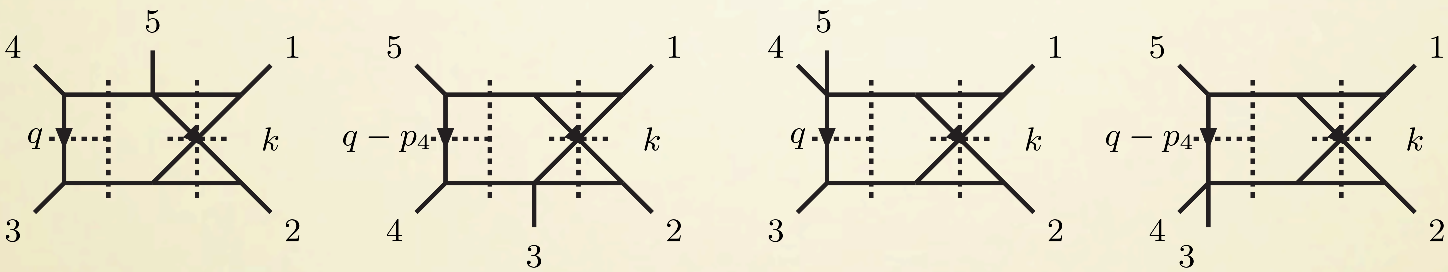
$$
\begin{aligned}
D_1 &= k^2\\
D_2 &= (k+p_2)^2\\
D_3 &= (k+q-p_4-p_5)^2\\
D_4 &= q^2\\
D_5 &= (q+p_3)^2\\
D_6 &= (q-p_4)^2\\
D_7 &= (q-p_4-p_5)^2\\
D_8 &= (q+k+p_2+p_3)^2\ .
\end{aligned}
$$



$N(q,k) = 2\,q\cdot v + \alpha$ Carrasco & Johansson (2011)

$$v^\mu = \frac{1}{4}\Big(\gamma_{12}(p_1^\mu - p_2^\mu) + \gamma_{23}(p_2^\mu - p_3^\mu) + 2\,\gamma_{45}(p_4^\mu - p_5^\mu) + \gamma_{13}(p_1^\mu - p_3^\mu)\Big)$$

$$\alpha = \frac{1}{4}\Big(2\,\gamma_{12}(s_{45}-s_{12}) + \gamma_{23}(s_{45}+3s_{12}-s_{13}) + 2\,\gamma_{45}(s_{14}-s_{15}) + \gamma_{13}(s_{12}+s_{45}-s_{13})\Big)$$



PentaCross Integrand Decomposition

$$
\begin{aligned}
D_1 &= k^2 \\
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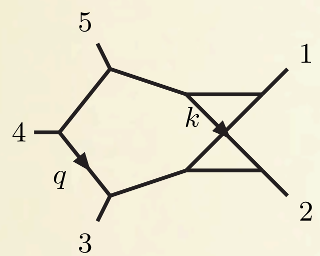
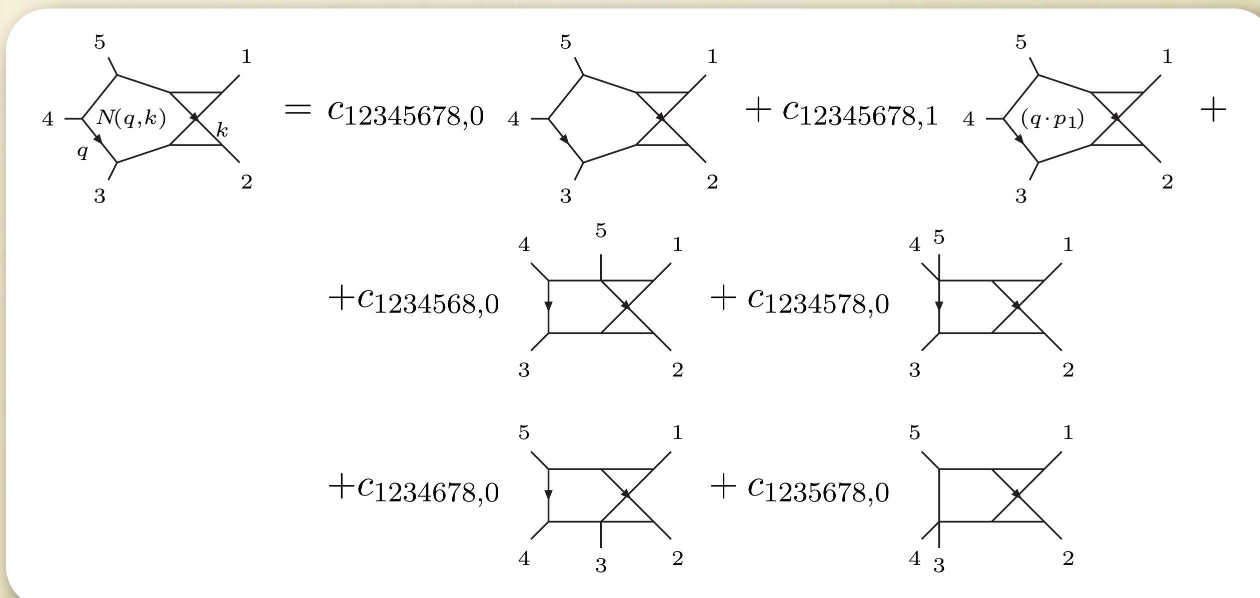
D_4 &= q^2 \\
D_5 &= (q+p_3)^2 \\
D_6 &= (q-p_4)^2 \\
D_7 &= (q-p_4-p_5)^2 \\
D_8 &= (q+k+p_2+p_3)^2 \ .
\end{aligned}
$$

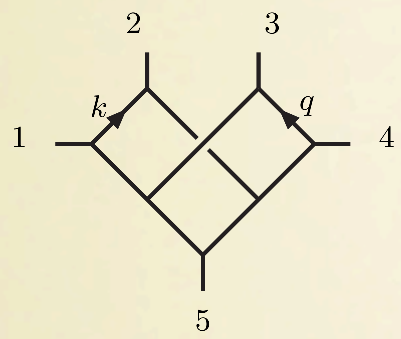


$$
\begin{aligned}
N(q,k) &= \Delta_{12345678}(q,k) + \\
&\quad +\Delta_{1234568}(q,k) D_7 + \Delta_{1234578}(q,k) D_6 + \\
&\quad +\Delta_{1234678}(q,k) D_5 + \Delta_{1235678}(q,k) D_4 = \\
&= c_{12345678,0} + c_{12345678,1}\ (q \cdot p_1) + \\
&\quad +c_{1234568,0} D_7 + c_{1234578,0} D_6 + \\
&\quad +c_{1234678,0} D_5 + c_{1235678,0} D_4 \ ,
\end{aligned}
$$



The coefficients are the same of the planar case.

The Last Contribution to the 5-point N=4 SYM



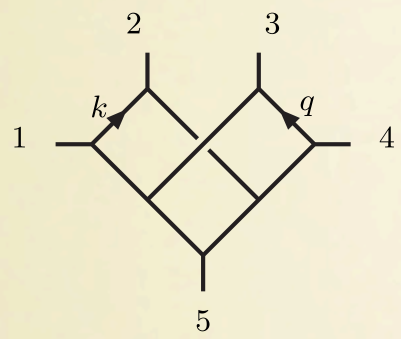
$$
\begin{aligned}
D_1 &= k^2 \\
D_2 &= (k-p_1)^2 \\
D_3 &= (k+p_2)^2 \\
D_4 &= q^2 \\
D_5 &= (q+p_3)^2 \\
D_6 &= (q-p_4)^2 \\
D_7 &= (q-k+p_1+p_3)^2 \\
D_8 &= (q-k-p_2-p_4)^2
\end{aligned}
$$

N(q,k) is ***linear*** in the loop momenta

Carrasco & Johansson (2011)

5-POINT 8FOLD-CUT

$D_1 = \ldots = D_8 = 0$ 8 solutions



$$\Delta_{12345678}(q,k) = \mathrm{Res}_{12345678}\left\{\mathcal{N}_{1\cdots8}(q,k)\right\} .$$

$$q^\mu = \sum_{i=1}^{4} y_i\, \tau_i^\mu\ , \qquad k^\mu = \sum_{i=1}^{4} x_i\, e_i^\mu\ .$$

$$e_1 = p_1, \qquad e_2 = p_2, \qquad \tau_1 = p_3, \qquad \tau_2 = p_4$$

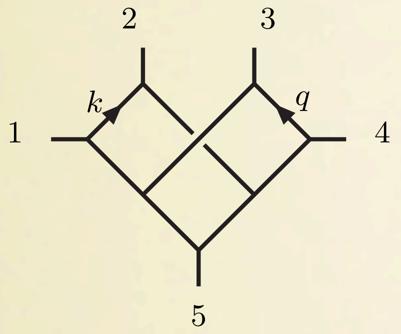
The residue contains 8 monomials $\{1, x_4, y_3, y_3^2, y_4, x_4 y_4, y_4^2, y_4^3\}$

[Maximum Cut Thm]

generic residue

5-point 8fold-cut

$D_1 = \ldots = D_8 = 0$ 8 solutions



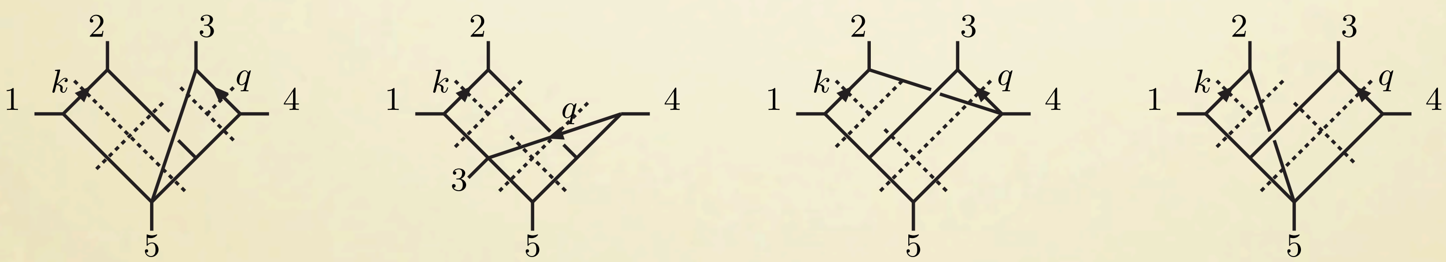
$$\Delta_{12345678}(q,k) = \mathrm{Res}_{12345678}\left\{\mathcal{N}_{1\cdots 8}(q,k)\right\} .$$

$$q^\mu = \sum_{i=1}^{4} y_i\, \tau_i^\mu\,, \qquad k^\mu = \sum_{i=1}^{4} x_i\, e_i^\mu\,.$$

$$e_1 = p_1, \qquad e_2 = p_2, \qquad \tau_1 = p_3, \qquad \tau_2 = p_4$$

The residue contains 8 monomials $\{1, x_4, y_3, y_3^2, y_4, x_4 y_4, y_4^2, y_4^3\}$

... further reduction ...



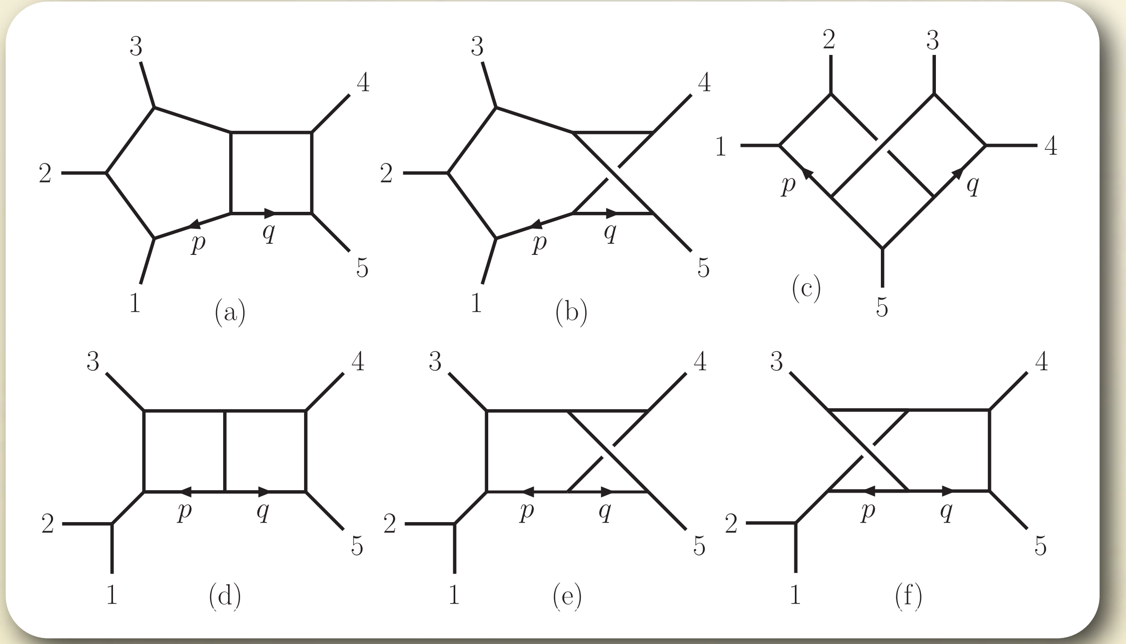
Complete Decomposition

Global (N=N)-test fulfilled!

2-loop 5-point amplitudes in N=8 SuGra

Same topologies as in the N=4 SYM, but N(q,k) is ***quadratic*** in the loop momenta

Carrasco & Johansson (2011)



The integrand reduction is analogous to the N=4 SYM case, involving the same cuts and residues.

Due to one extra power of loop momenta, the reduction involves also ***6-denominator diagrams***: in the corresponding residues, the constant term is the only non-vanishing coefficient.

Conclusions

- A unique mathematical framework for Amplitudes at any order in Perturbation Theory
  - one ingredient: Feynman denominator
  - one operation: *partial fractioning*
- Multivariate Polynomial Division/Groebner-basis generates the ***residue*** at an arbitrary cut
  - the general expression for the factorized amplitude
- Residues' ***classification*** complementary to Landau's singularity classification
- byproduct: the Maximum-cut Theorem
- *Recursive* generation of the *Integrand-decomposition Formula* @ any loop
- Amplitude decomposition from the shape of ***residues***
  - ISP's determine a (non-minimal) MI-set
  - application: planar and non-planar 2-loop 5-point N=4 SYM and N=8 SuGra (low-rank numerators w.r.t. QCD)

to appear soon

Let me thank my collaborators who, directly or indirectly, have provided the material for this lectures:

- Uli Schubert
- Hans van Deurzen
- Tiziano Peraro
- Gionata Luisoni
- Edoardo Mirabella
- Giovanni Ossola

Extra Slides

weak Nullstellensatz Theorem

Theorem 1.2.3 (Weak Hilbert Nullstellensatz). *If k is algebraically closed, then $V(S) = \emptyset$ iff there exists $f_1 \ldots f_N \in S$ and $g_1 \ldots g_N \in k[x_1, \ldots x_n]$ such that $\sum f_i g_i = 1$*

The German word nullstellensatz could be translated as "zero set theorem". The Weak Nullstellensatz can be rephrased as $V(S) = \emptyset$ iff $\langle S \rangle = (1)$. Since this result is central to much of what follows, **we will assume that k is alge-**

Radical Ideal

Given an ideal $\mathcal{J}$, the *radical* of $\mathcal{J}$ is $\sqrt{\mathcal{J}} \equiv \{f \in P[\mathbf{z}] : \exists\, s \in \mathbb{N},\ f^s \in \mathcal{J}\}$.

$\mathcal{J}$ is radical iff $\mathcal{J} = \sqrt{\mathcal{J}}$.

Finiteness Theorem

The following theorem bounds the number of points in $\mathbf{V}(I)$ whenever I is zero dimensional.

Theorem 3-4. *Let I be a zero-dimensional ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Then the number of points in $\mathbf{V}(I)$ is at most $\dim_{\mathbb{C}}(A)$. Equality occurs if and only if I is a radical ideal.*

Shape Lemma

since $p_{red}(x)/x$ is a cubic polynomial in x^2.

If I is a zero-dimensional radical ideal in $S = \mathbb{Q}[x_1, \ldots, x_n]$ then, possibly after a linear change of variables, the ring S/I is always isomorphic to the univariate quotient ring $\mathbb{Q}[x_i]/(I \cap \mathbb{Q}[x_i])$. This is the content of the following result.

Proposition 2.3. **(Shape Lemma)** *Let I be a zero-dimensional radical ideal in $\mathbb{Q}[x_1, \ldots, x_n]$ such that all d complex roots of I have distinct x_n-coordinates. Then the reduced Gröbner basis of I in the lexicographic term order has the shape*

$$\mathcal{G} \quad = \quad \{\, x_1 - q_1(x_n),\, x_2 - q_2(x_n),\, \ldots, x_{n-1} - q_{n-1}(x_n),\, r(x_n)\, \}$$

where r is a polynomial of degree d and the q_i are polynomials of degree $\leq d - 1$.

For polynomial systems of moderate size, `Singular` is ...

MCT: PROOF (PART 1)

Proof. Let us parametrize the propagators using 4ℓ variables $\mathbf{z} = (z_1, \ldots z_{4\ell})$. In this parametrization, the solutions of the maximum-cut read,

$$\mathbf{z}^{(i)} = \left(z_1^{(i)}, \ldots, z_{4\ell}^{(i)}\right) \ , \text{ with } i = 1, \ldots, n_s \ .$$

Let $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ be the ideal generated by the on-shell denominators, $\mathcal{J}_{i_1 \cdots i_{4\ell}} = \langle D_{i_1}, \ldots, D_{i_{4\ell}} \rangle$. According to the assumptions, the number n_s of the solutions is finite, and each of them has multiplicity one, therefore $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ is zero-dimensional and radical [1], In this case, the *Finiteness Theorem* ensures that the remainder of the division of any polynomial modulo $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ can be parametrised exactly by n_s coefficients.

MCT: PROOF (PART 2)

Moreover, up to a linear coordinate change, we can assume that all the solutions of the system have distinct first coordinate z_1, i.e. $z_1^{(i)} \neq z_1^{(j)} \ \forall \ i \neq j$. We observe that $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ and z_1 are in the *Shape Lemma* position therefore a Gröbner basis for the lexicographic order $z_1 < z_2 < \cdots < z_n$ is $\mathcal{G}_{i_1 \cdots i_{4\ell}} = \{g_1, \ldots, g_{4\ell}\}$, in the form

$$\begin{cases} g_1(\mathbf{z}) &= \quad f_1(z_1) \\ g_2(\mathbf{z}) &= \ z_2 - f_2(z_1) \\ & \vdots \\ g_{4\ell}(\mathbf{z}) &= z_{4\ell} - f_{4\ell}(z_1) \ . \end{cases}$$

The functions f_i are univariate polynomials in z_1. In particular f_1 is a rank-n_s square-free polynomial

$$f_1(z_1) = \prod_{i=1}^{n_s} \left(z_1 - z_1^{(i)} \right) \ ,$$

i.e. it does not exhibits repeated roots. The multivariate division of $\mathcal{N}_{i_1 \cdots 1_{4\ell}}$ modulo $\mathcal{G}_{i_1 \cdots i_{4\ell}}$ leaves a remainder $\Delta_{i_1 \cdots i_{4\ell}}$ which is a univariate polynomial in z_1 of degree $(n_s - 1)$ in accordance with the *Finiteness Theorem*. $\square$

QCD recursion relations from the largest time equation

Vaman, Yao (2005)

The factorization procedure is to cut these q_i successively by shifting them by $z\eta$. The on-shell conditions will give us a set of solutions, points in the complex plane, namely

$$z_i = \frac{q_i^2 + m_i^2}{2\eta \cdot q_i},$$

The identity which we want to establish is

$$\begin{aligned}
\frac{1}{q_1^2+m_1^2}\frac{1}{q_2^2+m_2^2}\cdots\frac{1}{q_{n-1}^2+m_{n-1}^2} &= \frac{1}{q_1^2+m_1^2}\frac{1}{(q_2-z_1\eta)^2+m_2^2}\cdots\frac{1}{(q_{n-1}-z_1\eta)^2+m_{n-1}^2} \\
&+ \frac{1}{(q_1-z_2\eta)^2+m_1^2}\frac{1}{q_2^2+m_2^2}\cdots\frac{1}{(q_{n-1}-z_2\eta)^2+m_{n-1}^2} \\
&+ \cdots\cdots\cdots \\
&+ \frac{1}{(q_1-z_{n-1}\eta)^2+m_1^2}\cdots\frac{1}{(q_{n-2}-z_{n-1}\eta)^2+m_{n-2}^2}\frac{1}{q_{n-1}^2+m_{n-1}^2}
\end{aligned}$$

making cuts, we have

$$\bar{q}_i^2 + m_i^2 = 0 \ \rightarrow \ q_i^2 + m_i^2 = 2z_i\eta \cdot q_i\,.$$

$$(q_i - z_j\eta)^2 + m_i^2 = q_i^2 + m_i^2 - 2z_j\eta \cdot q_i\,.$$

$$(q_i - z_j\eta)^2 + m_i^2 = 2\eta \cdot q_i(z_i - z_j)\,.$$

Putting these together, we see the identity holds if one can show

$$\begin{aligned}
\frac{(-1)^n}{z_1 z_2 \cdots z_{n-1}} &= \frac{1}{z_1(z_1-z_2)(z_1-z_3)\cdots(z_1-z_{n-1})} \\
&+ \frac{1}{(z_2-z_1)z_2(z_2-z_3)\cdots(z_2-z_{n-1})} \\
&\cdots\cdots\cdots \\
&+ \frac{1}{(z_{n-1}-z_1)(z_{n-1}-z_2)\cdots(z_{n-1}-z_{n-2})z_{n-1}}\,.
\end{aligned} \tag{7.9}$$

This is so, because (7.9) is just a formula of partial fractioning, or it is just a statement that the integral

$$\int \frac{dz}{z(z-z_1)(z-z_2)\cdots(z-z_{n-1})} = 0$$

for a complex variable z over a contour which encloses all the poles.

2.2.2 Quintuple cut

The residue of the quintuple-cut, $\bar{D}_i = \ldots = \bar{D}_m = 0$, defined as,

$$\Delta_{ijk\ell m}(\bar{q}) = \mathrm{Res}_{ijk\ell m}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}}\right\} = c_{5,0}^{(ijk\ell m)}\ \mu^2\ .$$

2.2.2 Quintuple cut

The residue of the quintuple-cut, $\bar{D}_i = \ldots = \bar{D}_m = 0$, defined as,

$$\Delta_{ijk\ell m}(\bar{q}) = \text{Res}_{ijk\ell m}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}}\right\} = c_{5,0}^{(ijk\ell m)} \, \mu^2 \, .$$

2.2.3 Quadruple cut

The residue of the quadruple-cut, $\bar{D}_i = \ldots = \bar{D}_\ell = 0$, defined as,

$$\Delta_{ijk\ell}(\bar{q}) = \text{Res}_{ijk\ell}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m}\right\}$$

$$= c_{4,0}^{(ijk\ell)} + c_{4,2}^{(ijk\ell)}\mu^2 + c_{4,4}^{(ijk\ell)}\mu^4 - \left(c_{4,1}^{(ijk\ell)} + c_{4,3}^{(ijk\ell)} \, \mu^2\right)\Big[(K_3 \cdot e_4)x_4 - (K_3 \cdot e_3)x_3\Big](e_1 \cdot e_2) \, ,$$

2.2.2 Quintuple cut

The residue of the quintuple-cut, $\bar{D}_i = \ldots = \bar{D}_m = 0$, defined as,

$$\Delta_{ijk\ell m}(\bar{q}) = \text{Res}_{ijk\ell m}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}}\right\} = c_{5,0}^{(ijk\ell m)}\ \mu^2\ .$$

2.2.3 Quadruple cut

The residue of the quadruple-cut, $\bar{D}_i = \ldots = \bar{D}_\ell = 0$, defined as,

$$\Delta_{ijk\ell}(\bar{q}) = \text{Res}_{ijk\ell}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m}\right\} = c_{4,0}^{(ijk\ell)} + c_{4,2}^{(ijk\ell)}\mu^2 + c_{4,4}^{(ijk\ell)}\mu^4 - \left(c_{4,1}^{(ijk\ell)} + c_{4,3}^{(ijk\ell)}\ \mu^2\right)\Big[(K_3\cdot e_4)x_4 - (K_3\cdot e_3)x_3\Big](e_1\cdot e_2)\,,$$

2.2.4 Triple cut

The residue of the triple-cut, $\bar{D}_i = \bar{D}_j = \bar{D}_k = 0$, defined as,

$$\Delta_{ijk}(\bar{q}) = \text{Res}_{ijk}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum_{i<<\ell}^{n-1} \frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell}\right\}$$

$$= c_{3,0}^{(ijk)} + c_{3,7}^{(ijk)}\mu^2 - \left((c_{3,1}^{(ijk)} + c_{3,8}^{(ijk)}\mu^2)x_4 + (c_{3,4}^{(ijk)} + c_{3,9}^{(ijk)}\mu^2)x_3\right)(e_1\cdot e_2)$$

$$+\left(c_{3,2}^{(ijk)}x_4^2 + c_{3,5}^{(ijk)}x_3^2\right)(e_1\cdot e_2)^2 - \left(c_{3,3}^{(ijk)}x_4^3 + c_{3,6}^{(ijk)}x_3^3\right)(e_1\cdot e_2)^3\ .$$

2.2.2 Quintuple cut

The residue of the quintuple-cut, $\bar{D}_i = \ldots = \bar{D}_m = 0$, defined as,

$$\Delta_{ijk\ell m}(\bar{q}) = \mathrm{Res}_{ijk\ell m}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}}\right\} = c_{5,0}^{(ijk\ell m)}\,\mu^2 \,.$$

2.2.3 Quadruple cut

The residue of the quadruple-cut, $\bar{D}_i = \ldots = \bar{D}_\ell = 0$, defined as,

$$\Delta_{ijk\ell}(\bar{q}) = \mathrm{Res}_{ijk\ell}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m}\right\} = c_{4,0}^{(ijk\ell)} + c_{4,2}^{(ijk\ell)}\mu^2 + c_{4,4}^{(ijk\ell)}\mu^4 - \left(c_{4,1}^{(ijk\ell)} + c_{4,3}^{(ijk\ell)}\,\mu^2\right)\Big[(K_3\cdot e_4)x_4 - (K_3\cdot e_3)x_3\Big](e_1\cdot e_2)\,,$$

2.2.4 Triple cut

The residue of the triple-cut, $\bar{D}_i = \bar{D}_j = \bar{D}_k = 0$, defined as,

$$\Delta_{ijk}(\bar{q}) = \mathrm{Res}_{ijk}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum_{i<<\ell}^{n-1} \frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell}\right\}$$

$$= c_{3,0}^{(ijk)} + c_{3,7}^{(ijk)}\mu^2 - \left((c_{3,1}^{(ijk)} + c_{3,8}^{(ijk)}\mu^2)x_4 + (c_{3,4}^{(ijk)} + c_{3,9}^{(ijk)}\mu^2)x_3\right)(e_1\cdot e_2) + \left(c_{3,2}^{(ijk)}x_4^2 + c_{3,5}^{(ijk)}x_3^2\right)(e_1\cdot e_2)^2 - \left(c_{3,3}^{(ijk)}x_4^3 + c_{3,6}^{(ijk)}x_3^3\right)(e_1\cdot e_2)^3 \,.$$

2.2.5 Double cut

The residue of the double-cut, $\bar{D}_i = \bar{D}_j = 0$, defined as,

$$\Delta_{ij}(\bar{q}) = \mathrm{Res}_{ij}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum_{i<<\ell}^{n-1} \frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell} - \sum_{i<<k}^{n-1} \frac{\Delta_{ijk}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k}\right\}$$

$$= c_{2,0}^{(ij)} + c_{2,9}^{(ij)}\mu^2 + \left(c_{2,1}^{(ij)}x_1 - c_{2,3}^{(ij)}x_4 - c_{2,5}^{(ij)}x_3\right)(e_1\cdot e_2) +$$

$$\left(c_{2,2}^{(ij)}x_1^2 + c_{2,4}^{(ij)}x_4^2 + c_{2,6}^{(ij)}x_3^2 - c_{2,7}^{(ij)}x_1x_4 - c_{2,8}^{(ij)}x_1x_3\right)(e_1\cdot e_2)^2 \,.$$

2.2.2 Quintuple cut

The residue of the quintuple-cut, $\bar{D}_i = \ldots = \bar{D}_m = 0$, defined as, $\Delta_{ijk\ell m}(\bar{q}) = \text{Res}_{ijk\ell m}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}}\right\} = c_{5,0}^{(ijk\ell m)}\ \mu^2$.

2.2.3 Quadruple cut

The residue of the quadruple-cut, $\bar{D}_i = \ldots = \bar{D}_\ell = 0$, defined as,

$$\Delta_{ijk\ell}(\bar{q}) = \text{Res}_{ijk\ell}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m}\right\} = c_{4,0}^{(ijk\ell)} + c_{4,2}^{(ijk\ell)}\mu^2 + c_{4,4}^{(ijk\ell)}\mu^4 - \left(c_{4,1}^{(ijk\ell)} + c_{4,3}^{(ijk\ell)}\ \mu^2\right)\Big[(K_3\cdot e_4)x_4 - (K_3\cdot e_3)x_3\Big](e_1\cdot e_2)\,,$$

2.2.4 Triple cut

The residue of the triple-cut, $\bar{D}_i = \bar{D}_j = \bar{D}_k = 0$, defined as,

$$\Delta_{ijk}(\bar{q}) = \text{Res}_{ijk}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum_{i<<\ell}^{n-1} \frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell}\right\}$$

$$= c_{3,0}^{(ijk)} + c_{3,7}^{(ijk)}\mu^2 - \left((c_{3,1}^{(ijk)} + c_{3,8}^{(ijk)}\mu^2)x_4 + (c_{3,4}^{(ijk)} + c_{3,9}^{(ijk)}\mu^2)x_3\right)(e_1\cdot e_2) + \left(c_{3,2}^{(ijk)}x_4^2 + c_{3,5}^{(ijk)}x_3^2\right)(e_1\cdot e_2)^2 - \left(c_{3,3}^{(ijk)}x_4^3 + c_{3,6}^{(ijk)}x_3^3\right)(e_1\cdot e_2)^3\ .$$

2.2.5 Double cut

2.2.6 Single cut

The residue of the single-cut, $\bar{D}_i = 0$, defined as,

$$\Delta_i(\bar{q}) = \text{Res}_i\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum_{i<<\ell}^{n-1} \frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell} - \sum_{i<<k}^{n-1} \frac{\Delta_{ijk}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k} - \sum_{i<j}^{n-1} \frac{\Delta_{ij}(\bar{q})}{\bar{D}_i\bar{D}_j}\right\}$$

$$= c_{1,0}^{(i)} + \left(c_{1,1}^{(i)}x_2 + c_{1,2}^{(i)}x_1 - c_{1,3}^{(i)}x_4 - c_{1,4}^{(i)}x_3\right)(e_1\cdot e_2)\ .$$

2.2.2 Quintuple cut

The residue of the quintuple-cut, $\bar{D}_i = \ldots = \bar{D}_m = 0$, defined as,

$$\Delta_{ijk\ell m}(\bar{q}) = \mathrm{Res}_{ijk\ell m}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}}\right\} = c_{5,0}^{(ijk\ell m)}\, \mu^2 \, .$$

2.2.3 Quadruple cut

The residue of the quadruple-cut, $\bar{D}_i = \ldots = \bar{D}_\ell = 0$, defined as,

$$\Delta_{ijk\ell}(\bar{q}) = \mathrm{Res}_{ijk\ell}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m}\right\} = c_{4,0}^{(ijk\ell)} + c_{4,2}^{(ijk\ell)}\mu^2 + c_{4,4}^{(ijk\ell)}\mu^4 - \left(c_{4,1}^{(ijk\ell)} + c_{4,3}^{(ijk\ell)}\,\mu^2\right)\Big[(K_3\cdot e_4)x_4 - (K_3\cdot e_3)x_3\Big](e_1\cdot e_2)\,,$$

2.2.4 Triple cut

The residue of the triple-cut, $\bar{D}_i = \bar{D}_j = \bar{D}_k = 0$, defined as,

$$\Delta_{ijk}(\bar{q}) = \mathrm{Res}_{ijk}\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum^{n-1} \frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell}\right\}$$

SAMURAI

Ossola, Reiter, Tramontano, & P.M. (2010)

Scattering AMplitudes from Unitarity-based Reduction Algorithm at the Integrand-level

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$$\Delta_{ij} \ldots \left\{\ldots \frac{}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m} \frac{}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum_{i<<\ell} \frac{}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell} - \sum_{i<<k} \frac{}{\bar{D}_i\bar{D}_j\bar{D}_k}\right\}$$

$$= c_{2,0}^{(ij)} + c_{2,9}^{(ij)}\mu^2 + \left(c_{2,1}^{(ij)}x_1 - c_{2,3}^{(ij)}x_4 - c_{2,5}^{(ij)}x_3\right)(e_1\cdot e_2) + \left(c_{2,2}^{(ij)}x_1^2 + c_{2,4}^{(ij)}x_4^2 + c_{2,6}^{(ij)}x_3^2 - c_{2,7}^{(ij)}x_1x_4 - c_{2,8}^{(ij)}x_1x_3\right)(e_1\cdot e_2)^2 \, .$$

2.2.6 Single cut

The residue of the single-cut, $\bar{D}_i = 0$, defined as,

$$\Delta_i(\bar{q}) = \mathrm{Res}_i\left\{\frac{N(\bar{q})}{\bar{D}_0 \cdots \bar{D}_{n-1}} - \sum_{i<<m}^{n-1} \frac{\Delta_{ijk\ell m}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell\bar{D}_m} - \sum_{i<<\ell}^{n-1} \frac{\Delta_{ijk\ell}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell} - \sum_{i<<k}^{n-1} \frac{\Delta_{ijk}(\bar{q})}{\bar{D}_i\bar{D}_j\bar{D}_k} - \sum_{i<j}^{n-1} \frac{\Delta_{ij}(\bar{q})}{\bar{D}_i\bar{D}_j}\right\}$$

$$= c_{1,0}^{(i)} + \left(c_{1,1}^{(i)}x_2 + c_{1,2}^{(i)}x_1 - c_{1,3}^{(i)}x_4 - c_{1,4}^{(i)}x_3\right)(e_1\cdot e_2) \, .$$

2.3 Amplitude and master integrals

$$\mathcal{A}_n = \sum_{i<j<k<\ell}^{n-1} \left\{ c_{4,0}^{(ijk\ell)} I_{ijk\ell}^{(d)} + \frac{(d-2)(d-4)}{4} c_{4,4}^{(ijk\ell)} I_{ijk\ell}^{(d+4)} \right\}$$
$$+ \sum_{i<j<k}^{n-1} \left\{ c_{3,0}^{(ijk)} I_{ijk}^{(d)} - \frac{(d-4)}{2} c_{3,7}^{(ijk)} I_{ijk}^{(d+2)} \right\}$$
$$+ \sum_{i<j}^{n-1} \left\{ c_{2,0}^{(ij)} I_{ij}^{(d)} + c_{2,1}^{(ij)} J_{ij}^{(d)} + c_{2,2}^{(ij)} K_{ij}^{(d)} - \frac{(d-4)}{2} c_{2,9}^{(ij)} I_{ij}^{(d+2)} \right\}$$
$$+ \sum_{i}^{n-1} c_{1,0}^{(i)} I_i^{(d)} ,$$

$$\int d^d\bar{q} \frac{\mu^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_\ell} = \frac{(d-2)(d-4)}{4} I_{ijk\ell}^{(d+4)} ,$$
$$\int d^d\bar{q} \frac{\mu^2}{\bar{D}_i\bar{D}_j\bar{D}_k} = -\frac{(d-4)}{2} I_{ijk}^{(d+2)} ,$$
$$\int d^d\bar{q} \frac{\mu^2}{\bar{D}_i\bar{D}_j} = -\frac{(d-4)}{2} I_{ij}^{(d+2)} ,$$
$$\int d^d\bar{q} \frac{\bar{q}\cdot e_2}{\bar{D}_i\bar{D}_j} = J_{ij}^{(d)} ,$$
$$\int d^d\bar{q} \frac{(\bar{q}\cdot e_2)^2}{\bar{D}_i\bar{D}_j} = K_{ij}^{(d)} .$$

$$\int \frac{d^4p}{(2\pi)^4} \int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} (\mu^2)^r f(p^\alpha,\mu^2) = \int \frac{d^4p}{(2\pi)^4} \int d\Omega_{-1-2\epsilon} \int_0^\infty \frac{d\mu^2}{2(2\pi)^{-2\epsilon}} (\mu^2)^{-1-\epsilon+r} f(p^\alpha,\mu^2)$$
$$= \frac{(2\pi)^{2r} \int d\Omega_{-1-2\epsilon}}{\int d\Omega_{2r-1-2\epsilon}} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^{2r-2\epsilon}\mu}{(2\pi)^{2r-2\epsilon}} f(p^\alpha,\mu^2)$$
$$= -\epsilon(1-\epsilon)(2-\epsilon)\cdots(r-1-\epsilon)\,(4\pi)^r \int \frac{d^{4+2r-2\epsilon}P}{(2\pi)^{4+2r-2\epsilon}} f(p^\alpha,\mu^2) ,$$

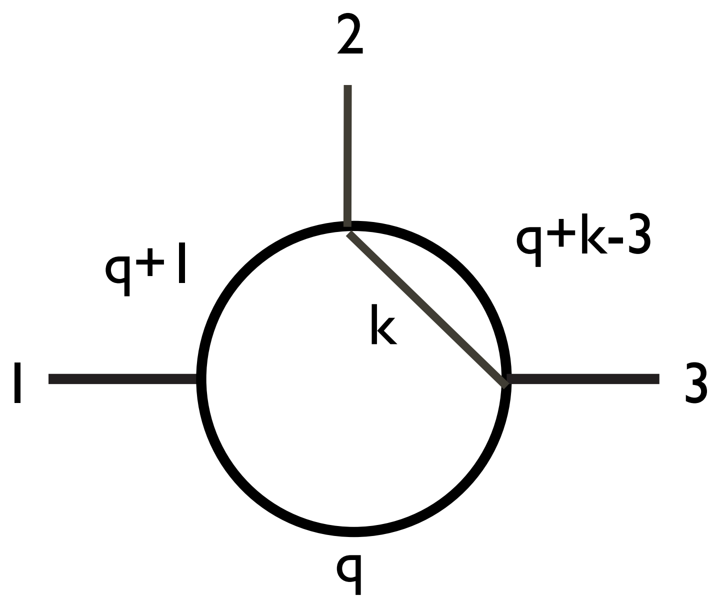
- **2-D Minkowski product**

```
MP[a_, b_] := a[[1]] * b[[1]] - a[[2]] * b[[2]] ;
```

- **Kinematics**

```
N[P11] = 0.3 * Sqrt[11] // N;
N[P12] = 0.5 * Sqrt[13] // N;
N[P21] = -0.1 * Sqrt[17] // N;
N[P22] = 0.7 * Sqrt[19] // N;

P1 = {P11, P12};
P2 = {P21, P22};
P3 = -P1 - P2;

MP[P1, P1] // N
MP[P2, P2] // N
MP[P3, P3] // N
```

```
-2.26

-9.14

-23.2219
```



2D Maximum Cut

- **orthogonal momentum: PT.P1=0**

```
PT = {P12, P11}
```

```
{P12, P11}
```

```
MP[PT, PT] // N
```

```
2.26
```

```
MP[P1, PT] // N
```

```
0.
```

Loop momenta decomposition

```
q = x1 * P1 + x2 * PT;
k = y1 * P1 + y2 * PT;
```

scalar products vs components

```
MP[q, P1] // N // Expand // Chop
```

$-2.26\ x1$

```
MP[q, PT] // N // Expand // Chop
```

$2.26\ x2$

```
MP[k, P1] // N // Expand // Chop
```

$-2.26\ y1$

```
MP[k, PT] // N // Expand // Chop
```

$2.26\ y2$

```
Vars = {x1, x2, y1, y2};
```

Denominators

```
D1 = MP[q, q];
D2 = MP[q + P1, q + P1];
D3 = MP[q + k - P3, q + k - P3];
D4 = MP[k, k];
```

```
Dens = {D1, D2, D3, D4} // N // Expand // Chop;
```

```
For[i = 1, i ≤ 4, i++,
 Print[Dens[[i]]];
]
```

$-2.26\ x1^2 + 2.26\ x2^2$

$-2.26 - 4.52\ x1 - 2.26\ x1^2 + 2.26\ x2^2$

$-23.2219 - 16.3419\ x1 - 2.26\ x1^2 - 7.55848\ x2 + 2.26\ x2^2 -$
$16.3419\ y1 - 4.52\ x1\ y1 - 2.26\ y1^2 - 7.55848\ y2 + 4.52\ x2\ y2 + 2.26\ y2^2$

$-2.26\ y1^2 + 2.26\ y2^2$

▪ On-shell Cut-conditions

```
Cut = Table[Dens[[i]] == 0, {i, 1, 4}]
```

$$\{-2.26\,x1^2 + 2.26\,x2^2 == 0,\ -2.26 - 4.52\,x1 - 2.26\,x1^2 + 2.26\,x2^2 == 0,\ -23.2219 - 16.3419\,x1 - 2.26\,x1^2 - 7.55848\,x2 + 2.26\,x2^2 - 16.3419\,y1 - 4.52\,x1\,y1 - 2.26\,y1^2 - 7.55848\,y2 + 4.52\,x2\,y2 + 2.26\,y2^2 == 0,\ -2.26\,y1^2 + 2.26\,y2^2 == 0\}$$

▪ Solutions

```
Sols = NSolve[Cut, Vars];

For[i = 1, i ≤ 4, i++,
 Print[Sols[[i]]];
]
```

{x1 → -0.5, x2 → -0.5, y1 → -2.64384, y2 → 2.64384}

{x1 → -0.5, x2 → 0.5, y1 → -2.14384, y2 → 2.14384}

{x1 → -0.5, x2 → 0.5, y1 → -0.971612, y2 → -0.971612}

{x1 → -0.5, x2 → -0.5, y1 → -0.471612, y2 → -0.471612}

```
Length[Sols]
```

4

- **Rank-5 numerator**

```
Num = (
  c[0] +
   Sum[c[i1] * Vars[[i1]], {i1, 1, Length[Vars]}] +
   Sum[c[i1, i2] * Vars[[i1]] * Vars[[i2]],
    {i1, 1, Length[Vars]}, {i2, i1, Length[Vars]}] +
   Sum[c[i1, i2, i3] * Vars[[i1]] * Vars[[i2]] * Vars[[i3]],
    {i1, 1, Length[Vars]}, {i2, i1, Length[Vars]}, {i3, i2, Length[Vars]}] +
   Sum[c[i1, i2, i3, i4] * Vars[[i1]] * Vars[[i2]] * Vars[[i3]] * Vars[[i4]],
    {i1, 1, Length[Vars]}, {i2, i1, Length[Vars]},
    {i3, i2, Length[Vars]}, {i4, i3, Length[Vars]}] +
   Sum[c[i1, i2, i3, i4, i5] * Vars[[i1]] * Vars[[i2]] * Vars[[i3]] *
     Vars[[i4]] * Vars[[i5]], {i1, 1, Length[Vars]}, {i2, i1, Length[Vars]},
    {i3, i2, Length[Vars]}, {i4, i3, Length[Vars]}, {i5, i4, Length[Vars]}]
 )
```

```
c[0] + x1 c[1] + x2 c[2] + y1 c[3] + y2 c[4] + x1^2 c[1, 1] + x1 x2 c[1, 2] + x1 y1 c[1, 3] +
 x1 y2 c[1, 4] + x2^2 c[2, 2] + x2 y1 c[2, 3] + x2 y2 c[2, 4] + y1^2 c[3, 3] + y1 y2 c[3, 4] +
 y2^2 c[4, 4] + x1^3 c[1, 1, 1] + x1^2 x2 c[1, 1, 2] + x1^2 y1 c[1, 1, 3] + x1^2 y2 c[1, 1, 4] +
 x1 x2^2 c[1, 2, 2] + x1 x2 y1 c[1, 2, 3] + x1 x2 y2 c[1, 2, 4] + x1 y1^2 c[1, 3, 3] +
 x1 y1 y2 c[1, 3, 4] + x1 y2^2 c[1, 4, 4] + x2^3 c[2, 2, 2] + x2^2 y1 c[2, 2, 3] +
 x2^2 y2 c[2, 2, 4] + x2 y1^2 c[2, 3, 3] + x2 y1 y2 c[2, 3, 4] + x2 y2^2 c[2, 4, 4] +
 y1^3 c[3, 3, 3] + y1^2 y2 c[3, 3, 4] + y1 y2^2 c[3, 4, 4] + y2^3 c[4, 4, 4] + x1^4 c[1, 1, 1, 1] +
 x1^3 x2 c[1, 1, 1, 2] + x1^3 y1 c[1, 1, 1, 3] + x1^3 y2 c[1, 1, 1, 4] + x1^2 x2^2 c[1, 1, 2, 2] +
 x1^2 x2 y1 c[1, 1, 2, 3] + x1^2 x2 y2 c[1, 1, 2, 4] + x1^2 y1^2 c[1, 1, 3, 3] + x1^2 y1 y2 c[1, 1, 3, 4] +
 x1^2 y2^2 c[1, 1, 4, 4] + x1 x2^3 c[1, 2, 2, 2] + x1 x2^2 y1 c[1, 2, 2, 3] + x1 x2^2 y2 c[1, 2, 2, 4] +
 x1 x2 y1^2 c[1, 2, 3, 3] + x1 x2 y1 y2 c[1, 2, 3, 4] + x1 x2 y2^2 c[1, 2, 4, 4] +
 x1 y1^3 c[1, 3, 3, 3] + x1 y1^2 y2 c[1, 3, 3, 4] + x1 y1 y2^2 c[1, 3, 4, 4] + x1 y2^3 c[1, 4, 4, 4] +
 x2^4 c[2, 2, 2, 2] + x2^3 y1 c[2, 2, 2, 3] + x2^3 y2 c[2, 2, 2, 4] + x2^2 y1^2 c[2, 2, 3, 3] +
 x2^2 y1 y2 c[2, 2, 3, 4] + x2^2 y2^2 c[2, 2, 4, 4] + x2 y1^3 c[2, 3, 3, 3] + x2 y1^2 y2 c[2, 3, 3, 4] +
 x2 y1 y2^2 c[2, 3, 4, 4] + x2 y2^3 c[2, 4, 4, 4] + y1^4 c[3, 3, 3, 3] + y1^3 y2 c[3, 3, 3, 4] +
 y1^2 y2^2 c[3, 3, 4, 4] + y1 y2^3 c[3, 4, 4, 4] + y2^4 c[4, 4, 4, 4] + x1^5 c[1, 1, 1, 1, 1] +
 x1^4 x2 c[1, 1, 1, 1, 2] + x1^4 y1 c[1, 1, 1, 1, 3] + x1^4 y2 c[1, 1, 1, 1, 4] +
 x1^3 x2^2 c[1, 1, 1, 2, 2] + x1^3 x2 y1 c[1, 1, 1, 2, 3] + x1^3 x2 y2 c[1, 1, 1, 2, 4] +
 x1^3 y1^2 c[1, 1, 1, 3, 3] + x1^3 y1 y2 c[1, 1, 1, 3, 4] + x1^3 y2^2 c[1, 1, 1, 4, 4] +
 x1^2 x2^3 c[1, 1, 2, 2, 2] + x1^2 x2^2 y1 c[1, 1, 2, 2, 3] + x1^2 x2^2 y2 c[1, 1, 2, 2, 4] +
 x1^2 x2 y1^2 c[1, 1, 2, 3, 3] + x1^2 x2 y1 y2 c[1, 1, 2, 3, 4] + x1^2 x2 y2^2 c[1, 1, 2, 4, 4] +
 x1^2 y1^3 c[1, 1, 3, 3, 3] + x1^2 y1^2 y2 c[1, 1, 3, 3, 4] + x1^2 y1 y2^2 c[1, 1, 3, 4, 4] +
 x1^2 y2^3 c[1, 1, 4, 4, 4] + x1 x2^4 c[1, 2, 2, 2, 2] + x1 x2^3 y1 c[1, 2, 2, 2, 3] +
 x1 x2^3 y2 c[1, 2, 2, 2, 4] + x1 x2^2 y1^2 c[1, 2, 2, 3, 3] + x1 x2^2 y1 y2 c[1, 2, 2, 3, 4] +
 x1 x2^2 y2^2 c[1, 2, 2, 4, 4] + x1 x2 y1^3 c[1, 2, 3, 3, 3] + x1 x2 y1^2 y2 c[1, 2, 3, 3, 4] +
 x1 x2 y1 y2^2 c[1, 2, 3, 4, 4] + x1 x2 y2^3 c[1, 2, 4, 4, 4] + x1 y1^4 c[1, 3, 3, 3, 3] +
 x1 y1^3 y2 c[1, 3, 3, 3, 4] + x1 y1^2 y2^2 c[1, 3, 3, 4, 4] + x1 y1 y2^3 c[1, 3, 4, 4, 4] +
 x1 y2^4 c[1, 4, 4, 4, 4] + x2^5 c[2, 2, 2, 2, 2] + x2^4 y1 c[2, 2, 2, 2, 3] + x2^4 y2 c[2, 2, 2, 2, 4] +
 x2^3 y1^2 c[2, 2, 2, 3, 3] + x2^3 y1 y2 c[2, 2, 2, 3, 4] + x2^3 y2^2 c[2, 2, 2, 4, 4] +
 x2^2 y1^3 c[2, 2, 3, 3, 3] + x2^2 y1^2 y2 c[2, 2, 3, 3, 4] + x2^2 y1 y2^2 c[2, 2, 3, 4, 4] +
 x2^2 y2^3 c[2, 2, 4, 4, 4] + x2 y1^4 c[2, 3, 3, 3, 3] + x2 y1^3 y2 c[2, 3, 3, 3, 4] +
 x2 y1^2 y2^2 c[2, 3, 3, 4, 4] + x2 y1 y2^3 c[2, 3, 4, 4, 4] + x2 y2^4 c[2, 4, 4, 4, 4] +
 y1^5 c[3, 3, 3, 3, 3] + y1^4 y2 c[3, 3, 3, 3, 4] + y1^3 y2^2 c[3, 3, 3, 4, 4] +
 y1^2 y2^3 c[3, 3, 4, 4, 4] + y1 y2^4 c[3, 4, 4, 4, 4] + y2^5 c[4, 4, 4, 4, 4]
```

```
Length[%]
```

```
126
```

Polynomial Division

```
GB = GroebnerBasis[Dens, Vars, MonomialOrder → DegreeLexicographic];

For[i = 1, i ≤ 4, i++,
 Print[GB[[i]]];
]
```

$0.5 + 1.\ x1$

$1.66493 + 0.836115\ x2 + 3.11545\ y1 + 1.\ y2^2$

$-0.960748 - 0.983196\ x2 - 1.67223\ y1 + 1.14906\ y2 + 1.\ y1\ y2$

$1.66493 + 0.836115\ x2 + 3.11545\ y1 + 1.\ y1^2$

```
remainder = PolynomialReduce[Num, GB, Vars, MonomialOrder → DegreeLexicographic][[2]];

Length[MonomialList[remainder, Vars]]
```

4

```
i = 1;
Collect[remainder, Vars, coeff[i++] &]
```

x2 coeff[1] + y1 coeff[2] + y2 coeff[3] + coeff[4]

Polynomial Division & Shape Lemma

```
GB = GroebnerBasis[Dens, Vars];

For[i = 1, i ≤ 4, i++,
 Print[GB[[i]]];
]
```

$2.5972 + 5.98633\, y2 - 0.7835\, y2^2 - 3.34446\, y2^3 + 1.\, y2^4$

$0.352643 + 1.\, y1 + 0.0586645\, y2 + 0.605742\, y2^2 - 0.113526\, y2^3$

$0.677279 + 1.\, x2 - 0.21859\, y2 - 1.06105\, y2^2 + 0.423009\, y2^3$

$0.5 + 1.\, x1$

```
remainder = PolynomialReduce[Num, GB, Vars][[2]];

Length[MonomialList[remainder, Vars]]
```

4

```
i = 1;
Collect[remainder, Vars, coeff[i++] &]
```

$y2^3\, coeff[1] + y2^2\, coeff[2] + y2\, coeff[3] + coeff[4]$