The Basic Structure of Scattering Amplitudes

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Alexander von Humboldt
Stiftung/Foundation
Sofja Kowaleskaja Award

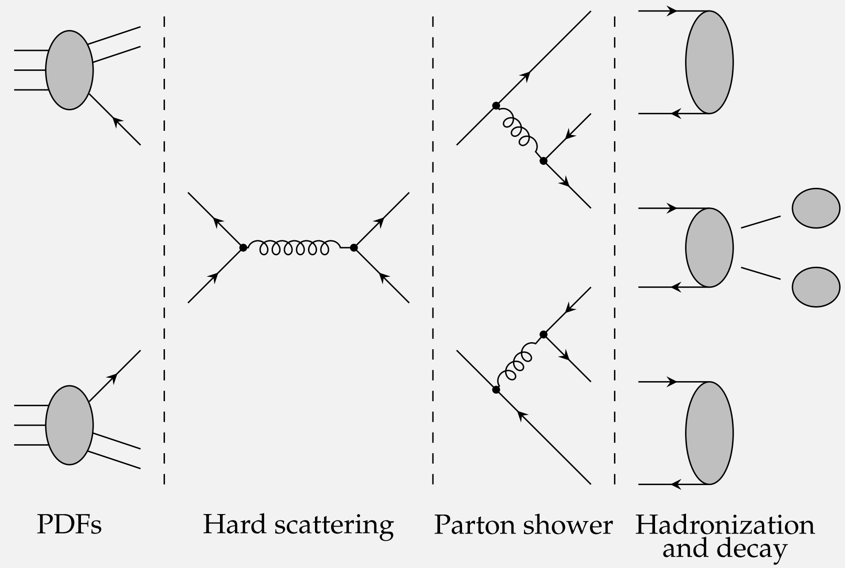
- LMU, Munich, 10-14 9.2012 :: Lecture I

  (these slides are supposed to be integrated with the blackboard notes)

References:

- L.J. Dixon, *Calculating Scattering Amplitudes efficiently*, hep-ph/9601359.
- A. Grozin, *Lectures on QED and QCD*, hep-ph/0508242
- M.E. Peskin, *Simplifying Multi-Jet QCD Computation*, arXiv:1101.2414 [hep-ph]
- M. Srednicki, Quantum Field Theory, web.physics.ucsb.edu/~mark/ms-qft-DRAFT.pdf
- Z. Trocsanyi, *Lectures on QFT*.

Illustration of hadron-hadron collision:



The cross section

The cross section is given by

$$\sigma = \frac{1}{2s} \int \mathrm{d}\phi_n \left(p_1, \ldots, p_n; Q\right) \frac{1}{S} \sum_{\text{spin}} \left|M_n\right|^2 , \tag{1.16}$$

where Q is the total incoming momentum, ($s = Q^2$) and

$$\mathrm{d}\phi_n = (2\pi)^4 \delta^d \left(Q^\mu - \sum_{j=1}^{n} p_j^\mu \right) \prod_{j=1}^{n} \frac{\mathrm{d}^d p_j}{(2\pi)^{d-1}} \delta_+ \left(p_j^2 - m_j^2\right)$$

is the phase space in $d = 4 - 2\epsilon$ dimensions (in reality $\epsilon \to 0$ such that $d = 4$, but we allow $d \neq 0$ for later purposes). The index $+$ of the δ-function means that we consider only positive solutions $E = +\sqrt{m^2 + \vec{p}^2}$, in other words

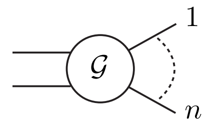
$$\delta_+ \left(p_j^2 - m_j^2\right) = \delta \left(p_j^2 - m_j^2\right) \theta\left(E\right),$$

where $\theta\left(E\right)$ is the Heaviside step-function.

S^{-1} comes from averaging for incoming (spin) states.

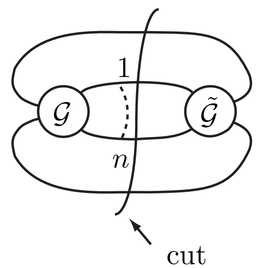
where $\theta(E)$ is the Heaviside step-function. The amplitude $\mathrm{i}M_n$ is obtained from all possible Feynman graphs and S^{-1} comes from averaging for incoming (spin) states. One can obtain $\sum |M_n|^2$ directly from the so called "cut" graphs following the Cutkosky rules. If a matrix element is given by the sum over all graphs $\mathcal{G}$,

$$\mathrm{i}\mathcal{M} = \sum\nolimits_{\mathcal{G}}$$



then the matrix element squared is given by the sum over all possible squared graph and over all possible cuts of these graphs:

$$\sum |\mathcal{M}_n|^2 = \sum\nolimits_{\text{cuts, } \mathcal{G}}$$



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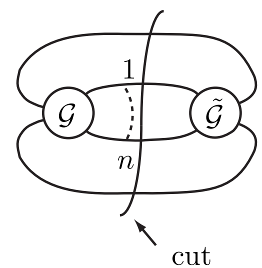


The Feynman rules for the cut graphs are the usual ones with the following additional rules:

1. the sign of explicit factors of $\mathrm{i} = \sqrt{-1}$ and directions of fermionic arrows and those of all momenta are reversed in $\tilde{\mathcal{G}}$ as compared to $\mathcal{G}$.

2. We do not integrate over the loop momentum of initial-state momenta,

3. A cut line j in the initial state means a factor of

   - $\not{p} + m_j$ if j is a fermion,
   - $\not{p} - m_j$ if j is a antifermion,
   - $-g_{\mu\nu}$ if j is a (massless) gauge boson.

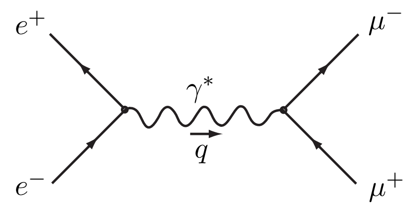
   In the final state the corresponding factors are

   - $\left(\not{p} \pm m_j\right) 2\pi\delta_+ \left(p_j^2 - m_j^2\right)$ if j is a fermion/antifermion,
   - $-g_{\mu\nu}\, 2\pi\, \delta_+ \left(p_j^2\right)$ if (massless) gauge boson.

   The δ_+ distributions express the on mass-shell conditions. These convert an integral over a loop momentum into the element of a one-particle phase-space measure.

Example: $e^+e^- \longrightarrow \mu^+\mu^-$

We consider as a very easy application of the Cutkosky rules the reaction $e^+e^- \rightarrow \mu^+\mu^-$. At the amplitude level there is only one Feynman graph that contributes at lowest order to this reaction:

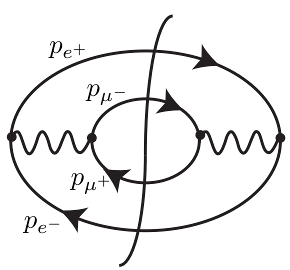


In order to better describe the kinematics of the reaction we use the Mandelstam variables known from the QFTI lecture,

$$
\begin{aligned}
s &= (p_{e^-} + p_{e^+})^2 = \left(p_{\mu^-} + p_{\mu^+}\right)^2 , \\
t &= \left(p_{e^-} - p_{\mu^-}\right)^2 = \left(p_{e^+} - p_{\mu^+}\right)^2 , \\
u &= \left(p_{e^-} - p_{\mu^+}\right)^2 = \left(p_{e^+} - p_{\mu^-}\right)^2 .
\end{aligned}
\tag{1.17}
$$

Furthermore we express the coupling in terms of $\alpha = \frac{e^2}{4\pi}$. The squared amplitude is then given by

$\sum_{\text{spin}} |\mathcal{M}_2|^2 =$



$$
\begin{aligned}
&= \tfrac{(4\pi\alpha)}{s^2}\,\mathrm{Tr}\left[\left(\not{p}_{e^+} - m_e\right)\gamma^\mu \left(\not{p}_{e^-} + m_e\right)\gamma^\nu\right] \\
&\quad\times \mathrm{Tr}\left[\left(\not{p}_{\mu^-} + m_\mu\right)\gamma_\mu \left(\not{p}_{\mu^+} - m_\mu\right)\gamma_\nu\right]
\end{aligned}
$$

Evaluating the traces[1] we obtain:

$$
\sum_{\text{spin}} |\mathcal{M}_2|^2 = 8\,(4\pi\alpha)^2 \left[\frac{u^2+t^2}{s^2} + 2\frac{m_\mu^2 - m_e^2}{s^2} - 8\frac{m_\mu^2 m_e^2}{s^2}\right] \tag{1.18}
$$

Spinor Formalism

We compute amplitudes of fixed helicity, which has the following advantages:

- Helicity is conserved along massless fermion lines.
- We can exploit gauge invariance and select an explicit representation for the polarization vectors.
- Different helicity configurations do not interfere, therefore, in computing $\sum_{\text{helicity}} |M_n|^2$, we sum the helicity amplitudes incoherently.

$$u_\pm(k) = \frac{1}{2}(1 \pm \gamma_5)u(k) \text{ and } v_\mp(k) = \frac{1}{2}(1 \pm \gamma_5)v(k)$$

$$\overline{u_\pm(k)} = \overline{u(k)}\frac{1}{2}(1 \mp \gamma_5) \text{ and } \overline{v_\mp(k)} = \overline{v(k)}\frac{1}{2}(1 \mp \gamma_5).$$

$$|i^\pm\rangle \equiv |k_i^\pm\rangle \equiv u_\pm(k_i) = v_\mp(k_i), \qquad \langle i^\pm| \equiv \langle k_i^\pm| \equiv \overline{u_\pm(k_i)} = \overline{v_\mp(k_i)}. \tag{11}$$

We define the basic spinor products by

$$\langle i\,j\rangle \equiv \langle i^-|j^+\rangle = \overline{u_-(k_i)}u_+(k_j), \qquad [i\,j] \equiv \langle i^+|j^-\rangle = \overline{u_+(k_i)}u_-(k_j). \tag{12}$$

$$\langle j\,l\rangle = -\langle l\,j\rangle, [j\,l] = -[l\,j], \qquad \langle j\,j\rangle = [j\,j] = 0.$$

The helicity projection implies that products like $\langle i^+|j^+\rangle$ vanish.

For numerical evaluation of the spinor products, it is useful to have explicit formulae for them, for some representation of the Dirac γ matrices. In the Dirac representation,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \tag{13}$$

the massless spinors can be chosen as follows,

$$u_+(k) = v_-(k) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{k^+} \\ \sqrt{k^-}e^{i\varphi_k} \\ \sqrt{k^+} \\ \sqrt{k^-}e^{i\varphi_k} \end{bmatrix}, \quad u_-(k) = v_+(k) = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{k^-}e^{-i\varphi_k} \\ -\sqrt{k^+} \\ -\sqrt{k^-}e^{-i\varphi_k} \\ \sqrt{k^+} \end{bmatrix}, \tag{14}$$

where

$$e^{\pm i\varphi_k} \equiv \frac{k^1 \pm ik^2}{\sqrt{(k^1)^2 + (k^2)^2}} = \frac{k^1 \pm ik^2}{\sqrt{k^+k^-}}, \qquad k^\pm = k^0 \pm k^3. \tag{15}$$

$$s_{ij} \;=\; (k_i + k_j)^2 \;=\; 2k_i \cdot k_j$$

$$\langle i\, j\rangle \,[j\, i] = \langle i^-|j^+\rangle\langle j^+|i^-\rangle = \mathrm{tr}\Big(\tfrac{1}{2}(1-\gamma_5)\, \not{k}_i\, \not{k}_j\Big) = 2k_i \cdot k_j = s_{ij}. \tag{18}$$

We also have the useful identities:
Gordon identity and projection operator:

$$\langle i^\pm|\gamma^\mu|i^\pm\rangle \;=\; 2k_i^\mu, \qquad\qquad |i^\pm\rangle\langle i^\pm| \;=\; \tfrac{1}{2}(1\pm\gamma_5)\, \not{k}_i \tag{19}$$

antisymmetry:

$$\langle j\, i\rangle = -\,\langle i\, j\rangle\,, \qquad [j\, i] = -\,[i\, j]\,, \qquad \langle i\, i\rangle = [i\, i] = 0 \tag{20}$$

Fierz rearrangement:

$$\langle i^+|\gamma^\mu|j^+\rangle\langle k^+|\gamma_\mu|l^+\rangle \;=\; 2\,[i\, k]\,\langle l\, j\rangle \tag{21}$$

charge conjugation of current:

$$\langle i^+|\gamma^\mu|j^+\rangle \;=\; \langle j^-|\gamma^\mu|i^-\rangle \tag{22}$$

Schouten identity:

$$\langle i\, j\rangle\,\langle k\, l\rangle \;=\; \langle i\, k\rangle\,\langle j\, l\rangle + \langle i\, l\rangle\,\langle k\, j\rangle\,. \tag{23}$$

In an n-point amplitude, momentum conservation, $\sum_{i=1}^n k_i^\mu = 0$, provides one more identity,

$$\sum_{\substack{i=1\\ i\neq j,k}}^{n} [j\, i]\,\langle i\, k\rangle \;=\; 0. \tag{24}$$

The next step is to introduce a spinor representation for the polarization vector for a massless gauge boson of definite helicity ± 1,

$$\varepsilon_\mu^\pm(k,q) \;=\; \pm\frac{\langle q^\mp|\gamma_\mu|k^\mp\rangle}{\sqrt{2}\langle q^\mp|k^\pm\rangle}\,, \tag{25}$$

where k is the vector boson momentum and q is an auxiliary massless vector, called the *reference momentum*, reflecting the freedom of on-shell gauge tranformations. We will not motivate Eq. 25, but just show that it has the desired properties. Since $\not{k}|k^\pm\rangle = 0$, $\varepsilon^\pm(k,q)$ is transverse to k, for any q,

$$\varepsilon^\pm(k,q)\cdot k \;=\; 0. \tag{26}$$

Complex conjugation reverses the helicity,

$$(\varepsilon_\mu^+)^* \;=\; \varepsilon_\mu^- \,. \tag{27}$$

The denominator gives ε_μ the standard normalization (using Eq. 21),

$$\begin{aligned}
\varepsilon^+\cdot(\varepsilon^+)^* \;&=\; \varepsilon^+\cdot\varepsilon^- \;=\; -\frac{1}{2}\frac{\langle q^-|\gamma_\mu|k^-\rangle\langle q^+|\gamma^\mu|k^+\rangle}{\langle q\,k\rangle\,[q\,k]} \;=\; -1,\\
\varepsilon^+\cdot(\varepsilon^-)^* \;&=\; \varepsilon^+\cdot\varepsilon^+ \;=\; \frac{1}{2}\frac{\langle q^-|\gamma_\mu|k^-\rangle\langle q^-|\gamma^\mu|k^-\rangle}{\langle q\,k\rangle^2} \;=\; 0.
\end{aligned} \tag{28}$$

the reference momentum q does amount to an on-shell gauge transformation, since ε_μ shifts by an amount proportional to k_μ:

$$\begin{aligned}
\varepsilon_\mu^+(\tilde{q})-\varepsilon_\mu^+(q) \;&=\; \frac{\langle\tilde{q}^-|\gamma_\mu|k^-\rangle}{\sqrt{2}\,\langle\tilde{q}\,k\rangle}-\frac{\langle q^-|\gamma_\mu|k^-\rangle}{\sqrt{2}\,\langle q\,k\rangle} = -\frac{\langle\tilde{q}^-|\gamma_\mu\not{k}|q^+\rangle+\langle\tilde{q}^-|\not{k}\gamma_\mu|q^+\rangle}{\sqrt{2}\,\langle\tilde{q}\,k\rangle\,\langle q\,k\rangle}\\
\;&=\; -\frac{\sqrt{2}\,\langle\tilde{q}\,q\rangle}{\langle\tilde{q}\,k\rangle\,\langle q\,k\rangle}\times k_\mu\,.
\end{aligned} \tag{29}$$

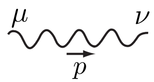
$$\mathrm{Soft}^{\mathrm{tree}}(a,s,b) \;=\; \frac{\langle a\,b\rangle}{\langle a\,s\rangle\,\langle s\,b\rangle}\,,$$

$$\sum_{\lambda=\pm}\varepsilon_\mu^\lambda(k,q)\,(\varepsilon_\nu^\lambda(k,q))^* \;=\; -\eta_{\mu\nu}+\frac{k_\mu q_\nu+k_\nu q_\mu}{k\cdot q}\,. \tag{30}$$

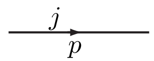
QED

Feynman rules in covariant Feynman gauge

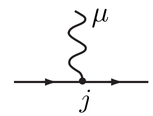
In the Feynman gauge the we have the following Feynman rules:



$$= \Delta_{\gamma_{\mu\nu}}(p) = -\mathrm{i}\frac{g_{\mu\nu}}{p^2}$$



$$= \Delta_j(p) = \mathrm{i}\,\frac{\not{p}+m}{p^2-m_j^2}$$



$$= \Gamma^{\mu}_{j\,f_j\bar{f}_j} = -\mathrm{i}e_j\,e\gamma^{\mu}$$

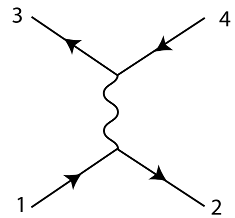
- outgoing fermion: $\bar{u}\,(p)$
- incoming fermion: $u\,(p)$
- outgoing photon: $\epsilon_{\mu}^{(\lambda)}\,(p)^*$
- outgoing antifermion: $v\,(p)$
- incoming antifermion: $\bar{v}\,(p)$
- incoming photon: $\epsilon_{\mu}^{(\lambda)}\,(p)$.

We introduce the following notation:

- external outgoing fermion of momentum p, helicity $\pm$: $\langle p\pm|$,
- external outgoing antifermion of momentum p, helicity $\pm$: $|p\mp\rangle$,
- external outgoing vector of momentum p, reference k, helicity $\pm$: $\epsilon_{\mu}^{\pm}\,(p,k) = \pm\frac{\langle p\pm|\gamma_\mu|k\pm\rangle}{\sqrt{2}\langle k\mp|p\pm\rangle}$.

The fermion propagator of momentum p in the direction of the fermion arrow is $+\mathrm{i}\frac{\not{p}}{p^2}$. The vector propagator of momentum p is: $-\frac{\mathrm{i}}{p^2}g_{\mu\nu}$.

$e^-_L e^+_R \;\to\; \mu^-_L \mu^+_R$ in QED.



$$q^2 \;=\; 2k_1 \cdot k_2 = \langle 12\rangle[21].$$

$$\begin{aligned} i\mathcal{M} &= (-ie)^2 \;\frac{-i}{q^2}\; \overline{U}_L(3)\gamma^\mu U_L(4)\,\overline{U}_L(2)\gamma_\mu U_L(1) \\ &= \frac{ie^2}{q^2}\langle 3\gamma^\mu 4]\langle 2\gamma_\mu 1] \\ &= \frac{2ie^2}{q^2}\langle 32\rangle[14]\;, \end{aligned}$$

$$i\mathcal{M} = 2ie^2\frac{\langle 32\rangle[14]\langle 32\rangle}{\langle 12\rangle[21]\langle 32\rangle}\;.$$

$$[21]\langle 32\rangle = [12]\langle 23\rangle = [1\;\not{2}\;3\rangle = [1(-\not{1}-\not{3}-\not{4})3\rangle$$

$$i\mathcal{M} = 2ie^2\frac{\langle 23\rangle^2}{\langle 12\rangle\langle 34\rangle}\;.$$

ets. Similarly, if we had multiplied instead by

$$\frac{[14]}{[14]}$$

a similar set of manipulations would have given

$$i\mathcal{M} = 2ie^2\frac{[14]^2}{[12][34]}\;,$$

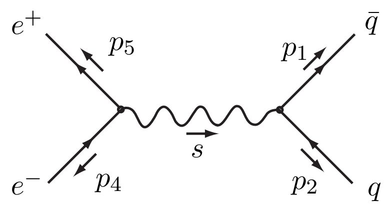
with square brackets only.

A simple application of the helicity formalism

We now compute the leading order contribution to the process $e^+e^- \rightarrow \mu^+\mu^-$ using the helicity formalism. Although it is not a physical choice, we consider the following crossing invariant kinematic configuration which is useful for obtaining the squared matrix element in either the annihilation or in the scattering channels:

$$0^\mu \rightarrow p_1^\mu + p_2^\mu + p_4^\mu + p_5^\mu. \tag{1.51}$$

As for the labelling convention, we anticipate further use of the result, when it will become clear. There is only one Feynman graph:



A general amplitude is given by

$$\mathcal{A}_4\left(1^{h_1}, 2^{h_2}, 4^{h_4}, 5^{h_5}\right) = e^2\, a_4\left(1^{h_1}, 2^{h_2}, 4^{h_4}, 5^{h_5}\right)$$

where the numbers are just a short-hand notation for the momenta, $j \equiv p_j$, which is a standard in the helicity formalism. Choosing a specific helicity configuration and applying the Feynman rules, we find

$$\begin{aligned} a_4\left(1^+, 2^-, 4^+, 5^-\right) &\equiv a_4\left(+,-,+,-\right) = \langle 1+|\, \mathrm{i}\gamma_\mu\, |2+\rangle \frac{(-\mathrm{i}g^{\mu\nu})}{s_{12}} \langle 5-|\, \mathrm{i}\gamma^\mu\, |4-\rangle \\ &= \mathrm{i}\, \frac{2\,[14]\,\langle 52\rangle}{\langle 12\rangle\,[21]} \frac{\langle 45\rangle}{\langle 45\rangle} = -\mathrm{i}\, \frac{2\,[12]\,\langle 25\rangle\,\langle 52\rangle}{[21]\,\langle 12\rangle\,\langle 45\rangle} = -\mathrm{i}\, \frac{2\,\langle 25\rangle^2}{\langle 12\rangle\,\langle 45\rangle}\,, \end{aligned} \tag{1.52}$$

where $s_{ij} = (p_i + p_j)^2$ and in the third line we used momentum conservation ($\not{4} = -\not{1} - \not{2} - \not{5}$) to rewrite $[14]\,\langle 45\rangle$ as

$$\begin{aligned} [14]\,\langle 45\rangle &= \left\langle 1+\left|\not{4}\right| 5+\right\rangle = \left\langle 1+\left|-\not{1} - \not{2} - \not{5}\right| 5+\right\rangle = \left\langle 1+\left|-\not{2}\right| 5+\right\rangle \\ &= -\,[12]\,\langle 25\rangle\,. \end{aligned} \tag{1.53}$$

The other helicity amplitudes can be obtained from that computed in Eq. (1.52), using discrete symmetry transformations of parity and charge conjugation. Parity transformation revereses all helicities of the helicity amplitude. It is implemented by the complex conjugation operation which substitutes $\langle ij\rangle \leftrightarrow [ji]$. Charge conjugation changes antifermions into fermions and vice versa, which in the present case amounts to interchanging indices 4 with 5 and/or 1 with 2. Thus,

$$\begin{aligned}
a_4\,(+,-,-,+) &= a_4\,(+,-,+,-)|_{4\leftrightarrow 5} \qquad \text{(from charge conjugation)} \\
&= -\mathrm{i}\,\frac{2\,\langle 24\rangle^2}{\langle 12\rangle\,\langle 54\rangle} \qquad (1.54) \\
a_4\,(-,+,-,+) &= a_4\,(+,-,+,-)|_{\langle ij\rangle\leftrightarrow[ji]} \qquad \text{(from parity transformation)} \\
&= -\mathrm{i}\,\frac{2\,[25]^2}{[12]\,[45]} \qquad (1.55) \\
a_4\,(-,+,+,-) &= -\mathrm{i}\,\frac{2\,[24]^2}{[12]\,[54]}\,. \qquad (1.56)
\end{aligned}$$

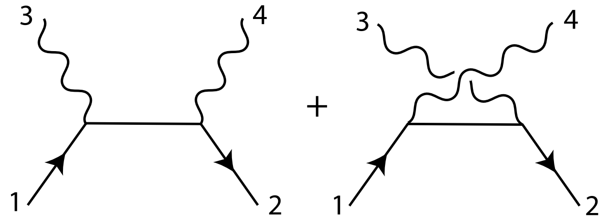
Computing the square of the amplitude, summed over helicities, we obtain

$$\sum_{\text{helicity}} |\mathcal{A}|^2 = e^4\,2\frac{4\left(s_{25}^2+s_{24}^2\right)}{s_{12}s_{45}} = 8\,(4\pi\alpha)^2\frac{t^2+u^2}{s^2}\,, \qquad (1.57)$$

where we adopted the usual notation of *Mandelstam*:

$$s_{12} = s_{45} = s\,, \qquad s_{24} = t\,, \qquad s_{25} = u\,.$$

amplitudes for $e^-_L e^+_R \to \gamma\gamma$



$$i\mathcal{M} = (-ie)^2 \langle 2 \left\{ \gamma \cdot \epsilon(4) \frac{i(2+4)}{s_{24}} \gamma \cdot \epsilon(3) + \gamma \cdot \epsilon(3) \frac{i(2+3)}{s_{23}} \gamma \cdot \epsilon(4) \right\} 1] \ .$$

There are four possible choices for the photon polarizations. However the cases $\gamma_R\gamma_L$ and $\gamma_L\gamma_R$ are related by interchange of the momenta 3 and 4. The cases $\gamma_R\gamma_R$ and $\gamma_L\gamma_L$ are related by parity, which interchanges states with R and L polarization.

Further, it is easy to see that the amplitudes for $\gamma_R\gamma_R$ and $\gamma_L\gamma_L$ are actually zero. For the case of $\gamma_R\gamma_R$, choose $r = 2$ for both polarization vectors,

$$\epsilon^\mu(3) = \frac{1}{\sqrt{2}} \frac{\langle 2\gamma^\mu 3]}{\langle 23\rangle} \ , \qquad \epsilon^\mu(4) = \frac{1}{\sqrt{2}} \frac{\langle 2\gamma^\mu 4]}{\langle 24\rangle} \ . \tag{40}$$

When these choices are used in (39), we find, with the use of the Fierz identity (18)

$$\langle 2\gamma \cdot \epsilon(4) \ \sim \ 2\langle 22\rangle[4 \ = \ 0 \ , \tag{41}$$

which vanishes because $\langle 22\rangle = 0$. A similar cancellation occurs with $\epsilon(3)$. So the entire matrix element vanishes. The amplitude for the case $\gamma_L\gamma_L$ must then also vanish by parity; alternatively, we can find the same cancellation for that case by using $r = 1$ in both polarization vectors.

To compute the amplitude for the case $\gamma_R \gamma_L$, choose

$$\epsilon^\mu(3) = \frac{1}{\sqrt{2}} \frac{\langle 2\gamma^\mu 3]}{\langle 23\rangle} \ , \qquad \epsilon^\mu(4) = -\frac{1}{\sqrt{2}} \frac{[1\gamma^\mu 4\rangle}{[14]} \ . \tag{42}$$

Then the second diagram in Fig. 3 vanishes by the logic of the previous paragraph. Using the Fierz identity, the first diagram gives

$$\begin{aligned} i\mathcal{M} &= \frac{-ie^2}{s_{24}} \frac{2\cdot 2}{(-2)\langle 23\rangle[14]} \langle 24\rangle[1(2+4)2\rangle[31] \\ &= \frac{2ie^2}{s_{13}\langle 23\rangle[14]} \langle 24\rangle[14]\langle 42\rangle[31] \\ &= \frac{2ie^2}{\langle 13\rangle[31]\langle 23\rangle[14]} \langle 24\rangle[14]\langle 42\rangle[31] \\ &= 2ie^2 \frac{(\langle 24\rangle)^2}{\langle 23\rangle\langle 31\rangle} \end{aligned} \tag{43}$$

QCD

Here we shall discuss how to calculate colour diagrams for $SU(N_c)$ colour group (in the Nature, quarks have $N_c = 3$ colours). For more details, see [4].

Elements of $SU(N_c)$ are complex $N_c \times N_c$ matrices U which are unitary ($U^+U = 1$) and have $\det U = 1$. Complex N_c-component column vectors q^i transforming as

$$q \to Uq \quad \text{or} \quad q^i \to U^i{}_j q^j \tag{A.1}$$

form the space in which the fundamental representation operates. Complex conjugate row vectors $q_i^+ = (q^i)^*$ transform as

$$q^+ \to q^+U^+ \quad \text{or} \quad q_i^+ \to q_j^+ (U^+)^j{}_i \quad \text{where} \quad (U^+)^j{}_i = \left(U^i{}_j\right)^* ; \tag{A.2}$$

this is the conjugated fundamental representation. The scalar product is invariant: $q^+q' \to q^+U^+Uq' = q^+q'$. In other words,

$$\delta^i_j \to \delta^k_l U^i{}_k (U^+)^l{}_j = U^i{}_k (U^+)^k{}_j = \delta^i_j \tag{A.3}$$

is an invariant tensor (it is the colour structure of a meson). The unit antisymmetric tensors $\varepsilon^{i_1 \ldots i_{N_c}}$ and $\varepsilon_{i_1 \ldots i_{N_c}}$ are also invariant:

$$\varepsilon^{i_1 \ldots i_{N_c}} \to \varepsilon^{j_1 \ldots j_{N_c}} \, U^{i_1}{}_{j_1} \cdots U^{i_{N_c}}{}_{j_{N_c}} = \det U \cdot \varepsilon^{i_1 \ldots i_{N_c}} = \varepsilon^{i_1 \ldots i_{N_c}} \tag{A.4}$$

(they are the colour structures of baryons and antibaryons).

Infinitesimal transformations are given by

$$U = 1 + i\alpha^a t^a \,, \tag{A.5}$$

where α^a are infinitesimal real parameters, and t^a are called generators (of the fundamental representation). They have the following properties:

$$\begin{aligned} U^+U &= 1 + i\alpha^a \left(t^a - (t^a)^+\right) = 1 \quad &\Rightarrow \quad (t^a)^+ = t^a \,, \\ \det U &= 1 + i\alpha^a \operatorname{Tr} t^a = 1 \quad &\Rightarrow \quad \operatorname{Tr} t^a = 0 \,, \end{aligned} \tag{A.6}$$

and are normalized by

$$\operatorname{Tr} t^a t^b = T_F \delta^{ab} \,; \tag{A.7}$$

usually, $T_F = 1/2$ is used, but we shall not specialize it. The space of unitary matrices is N_c^2-dimensional, and that of traceless unitary matrices — $(N_c^2 - 1)$-dimensional. Therefore, there are $N_c^2 - 1$ generators t^a which form a basis of this space. Their commutators are i times unitary traceless matrices, therefore,

$$[t^a, t^b] = i f^{abc} t^c \,, \tag{A.8}$$

where f^{abc} are real constants.

The quantities

$$A^a = q^+ t^a q'$$

transform as

$$A^a \to q^+ U^+ t^a U q' = U^{ab} A^b \,; \tag{A.9}$$

this is the adjoint representation. It is defined by

$$U^+ t^a U = U^{ab} t^b \,, \tag{A.10}$$

and hence

$$U^{ab} = \frac{1}{T_F} \operatorname{Tr} U^+ t^a U t^b \,. \tag{A.11}$$

The components $(t^a)^i{}_j$ are some fixed numbers; in other words, they form an invariant tensor (see (A.10)):

$$(t^a)^i{}_j \to U^{ab} U^i{}_k (t^b)^k{}_l (U^+)^l{}_j = (t^a)^i{}_j \,. \tag{A.12}$$

For an infinitesimal transformation,

$$A^a \to q^+ (1 - i\alpha^c t^c) t^a (1 + i\alpha^c t^c) q' = q^+ (t^a + i\alpha^c i f^{acb} t^b) q' \,,$$

so that

$$U^{ab} = \delta^{ab} + i\alpha^c (t^c)^{ab} \,, \tag{A.13}$$

where the generators in the adjoint representation are

$$(t^c)^{ab} = i f^{acb} \,. \tag{A.14}$$

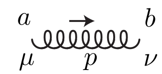
As for any representation, these generators satisfy the commutation relation

$$(t^a)^{dc} (t^b)^{ce} - (t^b)^{dc} (t^a)^{ce} = i f^{abc} (t^c)^{de} \,; \tag{A.15}$$

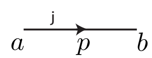
it follows from the Jacobi identity

$$\begin{aligned} &[t^a, [t^b, t^d]] + [t^b, [t^d, t^a]] + [t^d, [t^a, t^b]] = 0 \\ &= \left(i f^{bdc} i f^{ace} + i f^{dac} i f^{bce} + i f^{abc} i f^{dce} \right) t^e \,. \end{aligned}$$

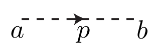
QCD



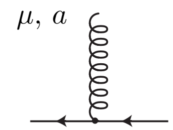
$\equiv \Delta^{a\,b}_{g,\,\mu\nu}\;(p) = \delta^{a\,b}\,\Delta_{\gamma,\,\mu\nu}\,(p)$



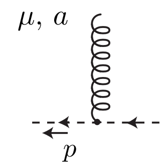
$\equiv \Delta^{a\,b}_{j}\;(p) = \delta^{a\,b}\,\Delta_{j}\,(p)$



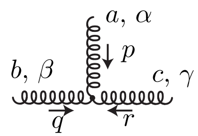
$\equiv \Delta^{a\,b}\;(p) = \delta^{a\,b}\,\frac{i}{p^2}$



$\equiv \Gamma^{\mu,\,a}_{gq\bar{q}} = -\mathrm{i}g_S\,t^a\gamma^\mu$

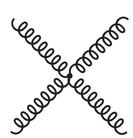


$\equiv \Gamma^{\mu,\,a}_{g\eta\bar{\eta}} = -\mathrm{i}g_S\,F^a\,p^\mu$



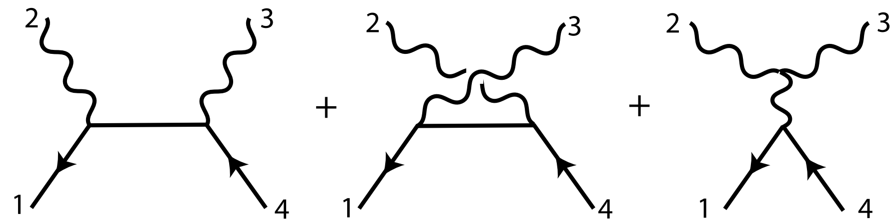
$\equiv \Gamma^{a,\,b,\,c}_{\alpha,\,\beta,\,\gamma}\,(p,q,r) = -\mathrm{i}g_s\,(F^a)_{bc}\;V_{\alpha,\,\beta,\,\gamma}\,(p,q,r) \qquad p+q+r=0$

$$V_{\alpha,\,\beta,\,\gamma}\,(p,q,r) = (p-q)_\gamma\,g_{\alpha\beta} + (q-r)_\alpha\,g_{\beta\gamma} + (r-p)_\beta\,g_{\alpha\gamma}$$



$\equiv \Gamma^{a,b,c,d}_{\alpha,\,\beta,\,\gamma,\,\delta}$

$q_L\overline{q}_R \to gg.$



$$i\mathcal{M} = (ig)^2\langle 1\left\{\gamma\cdot\epsilon(2)\frac{i(1+2)}{s_{12}}t^at^b\gamma\cdot\epsilon(3) + \gamma\cdot\epsilon(3)\frac{i(1+3)}{s_{13}}t^bt^a\gamma\cdot\epsilon(2)\right\}4]$$
$$+(ig)(-gf^{abc}t^c)\frac{-i}{s_{14}}\langle 1\gamma^\lambda 4]\cdot\Big(\epsilon(2)\cdot\epsilon(3)(2-3)_\lambda\ +\epsilon_\lambda(3)(2\cdot 3+2)\cdot\epsilon(2)+\epsilon_\lambda(2)(-2\cdot 2-3)\cdot\epsilon(3)\Big)$$

In this formula, t^a and t^b are the color $SU(3)$ representation matrices coupling to the gluons 2 and 3, respectively. The third diagram in Fig. 4 has a color structure that can be brought into the forms seen in the first two diagrams by writing

$$-gf^{abc}t^c = (ig)\cdot if^{abc}t^c = (ig)(t^at^b - t^bt^a)\ . \tag{46}$$

We will find it convenient to rescale the color matrices: $T^a = \sqrt{2}t^a$, so that the T^a are normalized to

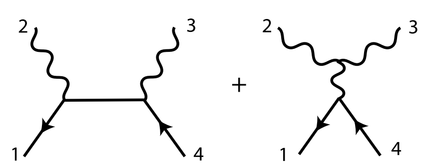
$$\mathrm{tr}[T^aT^b] = \delta^{ab}\ . \tag{47}$$

We can then write the amplitude in (45) in the form

$$i\mathcal{M} = i\mathbf{M}(1234)\cdot T^aT^b + i\mathbf{M}(1324)\cdot T^bT^a\ , \tag{48}$$

with

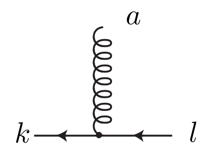
$$i\mathbf{M}(1234) = (\frac{ig}{\sqrt{2}})^2\Big[\langle 1\gamma\cdot\epsilon(2)\frac{i(1+2)}{s_{12}}\gamma\cdot\epsilon(3)4]\ +\frac{-i}{s_{23}}\langle 1\gamma^\lambda 4][\epsilon(2)\cdot\epsilon(3)(2-3)_\lambda\ +\epsilon_\lambda(3)(2\cdot 3+2)\cdot\epsilon(2)+\epsilon_\lambda(2)(-2\cdot 2-3)\cdot\epsilon(3)]\ .$$



The elements $i\mathbf{M}$ are called *color-ordered amplitudes.* the color-ordered amplitudes must be separately gauge-invariant.

Basics of colour algebra

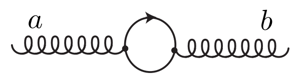
We now forget about all but the colour part of the Feynman rules and try first to develop an efficient technique to compute the coefficients involving the colour structure. This is possible because the colour degrees of freedom factorize from the other degrees of freedom completely. We use the following graphical representation for the colour charges in the fundamental representation:



$$= (t^a)_{kl}.$$

The normalization of these matrices is given by

$$\mathrm{Tr}\left(t^a t^b\right) \equiv$$

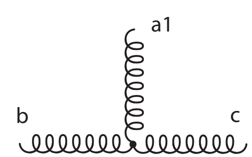


$$= T_R\,\delta^{ab} \equiv T_R \cdot$$

The ususal choice is $T_R = \frac{1}{2}$, but $T_R = 1$ is also used often. We shall use both.

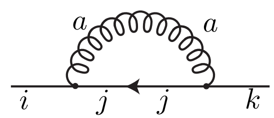
In the adjoint representation the colour charge T^a is represented by the matrix $(F^a)_{bc}$ that is related to the structure constants by

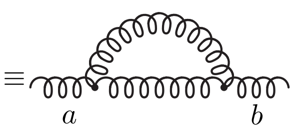
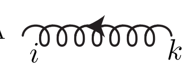
$$(F^a)_{bc} = \left(F^b\right)_{ca} = (F^c)_{ab} = -\mathrm{i}\, f_{abc} =$$



where F^a with $a = 1, \ldots, (N_c^2 - 1)$ are $\left(N_c^2 - 1\right) \times \left(N_c^2 - 1\right)$ matrices which again satisfie the commutation relation (1.1). The graphical notation in the adjoint representation is not unique. For the matrix $(F^a)_{bc}$ we assume an arrow pointing from index c to b, opposite to which we read the indices of (F^a). On the structure constans the indices are not distingushed, therefore arrow do not appear. However, these are completely antisymmetric in their indices, therefore, the ordering matters. By convention, in the graphical representation, the ordering of the indices is anticlockwise. The representation matrices are invariant under $SU(N)$ transformations.

The sums $\sum_a t^a_{ij} t^a_{jk}$ and $\mathrm{Tr}\big(F^a F^b\big)$ have two free indices in the fundamental and adjoint representation, respectively. These are invariant under $SU(N)$ transformations, therefore, must be proportional to the unit matrix,

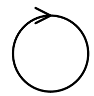
$\sum_a t^a_{ij} t^a_{jk} \equiv$  $= C_{\mathrm{F}}$  $= C_{\mathrm{F}}\, \delta_{ik}$

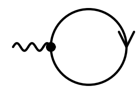
$\mathrm{Tr}\big(F^a F^b\big) \equiv$  $= C_{\mathrm{A}}$  $= C_{\mathrm{A}}\, \delta^{ab}$

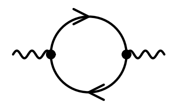
where C_{F} and C_{A} are the eigenvalues of the quadratic Casimir operator in the fundamental, respectively adjoint, representation. In the familiar case of angular momentum operator algebra ($SU(2)$), the quadratic Casimir operator is the square of the angular momentum with eigenvalues $j(j+1)$. The fundamental representation is two dimensional, realized by the (half of the) Pauli matrices acting on two-component spinors, when $j = 1/2$ and $C_{\mathrm{F}} = 1/2(1/2+1) = 3/4$. In the adjoint representation $j = 1$ and $C_{\mathrm{A}} = 2$. Below we derive the corrseponding values for general $SU(N)$.

Chromatica

It is very convenient to do colour calculations in graphical form [4]. Quark lines mean δ^i_j, gluon lines mean δ^{ab}, and quark-gluon vertices mean $(t^a)^i{}_j$. There is no need to invent names for indices; it is much easier to see which indices are contracted — they are connected by a line. Here are the properties of the generators t^a which we already know:

$\mathrm{Tr}\,1 = N_c$ or  $= N_c\,,$

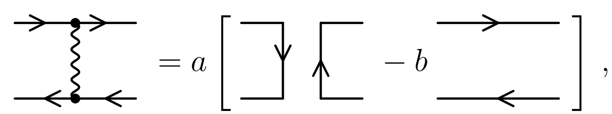
$\mathrm{Tr}\,t^a = 0$ or  $= 0\,,$ (A.16)

$\mathrm{Tr}\,t^a t^b = T_F \delta^{ab}$ or  $= T_F$  .

There is a simple and systematic method for calculation of colour factors — Cvitanović algorithm [4]. Now we are going to derive its main identity. The tensor $(t^a)^i{}_j (t^a)^k{}_l$ is invariant, because $(t^a)^i{}_j$ is invariant. It can be expressed via δ^i_j, the only independent invariant tensor with fundamental-representation indices (it is clear that $\varepsilon^{i_1 \ldots i_{N_c}}$ and $\varepsilon_{i_1 \ldots i_{N_c}}$ cannot appear in this expression, except the case $N_c = 2$; in this case $\varepsilon^{ik}\varepsilon_{jl}$ can appear, but it is expressible via δ^i_j). The general form of this expression is

$$(t^a)^i{}_j (t^a)^k{}_l = a \left[\delta^i_l \delta^k_j - b \delta^i_j \delta^k_l \right] , \tag{A.17}$$

or graphically

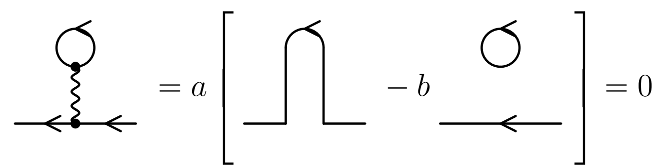


(A.18)

where a and b are some unknown coefficients. If we multiply (A.17) by δ^j_i,

$$(t^a)^i{}_i (t^a)^k{}_l = 0 = a \left[\delta^k_l - b N_c \delta^k_l \right] ,$$

i. e., close the upper line in (A.18),



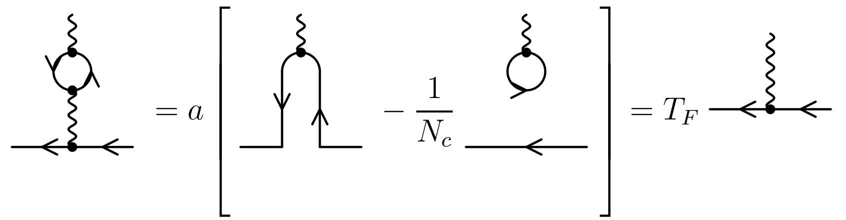
we obtain

$$b = \frac{1}{N_c} .$$

If we multiply (A.17) by $(t^b)^j{}_i$,

$$(t^b)^j{}_i (t^a)^i{}_j (t^a)^k{}_l = T_F (t^a)^k{}_l = a \left[(t^b)^k{}_l - \frac{1}{N_c} (t^b)^i{}_i \delta^k_l \right] ,$$

i. e., close the upper line in (A.18) onto a gluon,



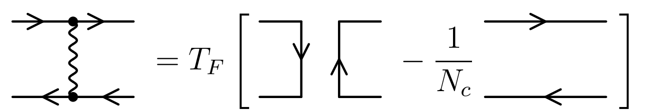
we obtain

$$a = T_F \, .$$

The final result is

$$(t^a)^i{}_j (t^a)^k{}_l = T_F \left[\delta^i_l \delta^k_j - \frac{1}{N_c} \delta^i_j \delta^k_l \right] , \tag{A.19}$$

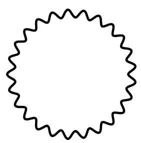
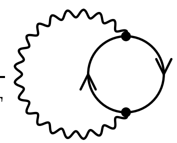
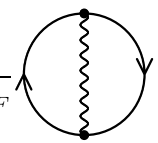
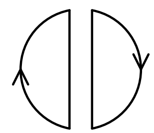
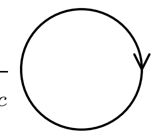
or graphically



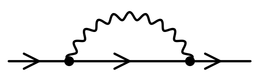
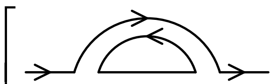
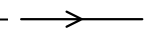
(A.20)

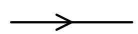
This identity allows one to eliminate a gluon exchange in a colour diagram: such an exchange is replaced by the exchange of a quark–antiquark pair, from which its colour-singlet part is subtracted.

The Cvitanović algorithm consists of elimination gluon exchanges (A.20) and using simple rules (A.16). Let's consider a simple application: counting gluon colours. Their number is

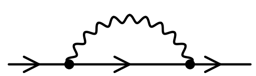
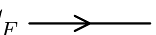
$N_g =$  $= \dfrac{1}{T_F}$  $= \dfrac{1}{T_F}$  $=$  $- \dfrac{1}{N_c}$  $= N_c^2 - 1\,,$

Now we consider a very important example:

 $= T_F \Bigg[$  $- \dfrac{1}{N_c}$  $\Bigg]$

$= T_F \left(N_c - \dfrac{1}{N_c} \right)$  .

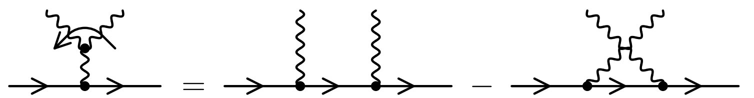
The result is

 $= C_F$  or $t^a t^a = C_F\,,$ (A.23)

where the Casimir operator in the fundamental representation is

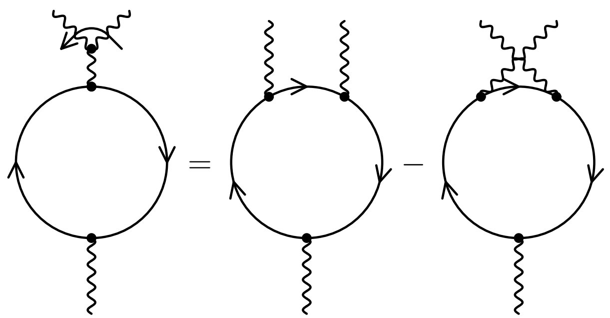
$$C_F = T_F \left(N_c - \frac{1}{N_c} \right) . \tag{A.24}$$

Colour diagrams can contain one more kind of elements: 3–gluon vertices if^{abc}. The definition (A.8) when written graphically is

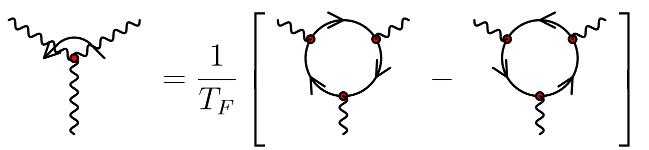


(A.25)

Let's close the quark line onto a gluon:



Therefore,

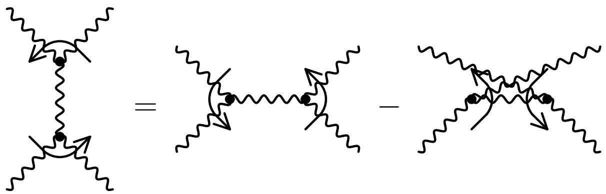


(A.26)

This is the final rule of the Cvitanović algorithm: elimination of 3–gluon vertices.

$$[t^a, t^b] = if^{abc}t^c\,, \tag{A.8}$$

The commutation relation (A.15) can be rewritten graphically, similarly to (A.25):

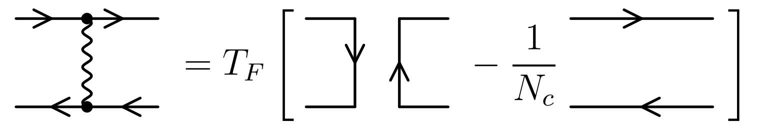


$$.\qquad\text{(A.27)}$$

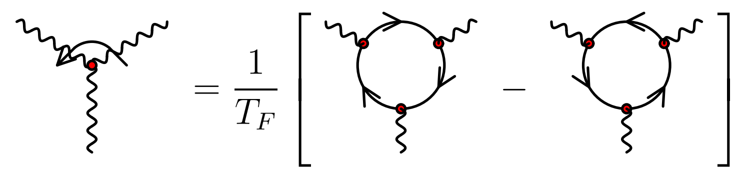
Jacobi identity

$$[t^a,[t^b,t^d]]+[t^b,[t^d,t^a]]+[t^d,[t^a,t^b]]=0$$
$$=\left(if^{bdc}if^{ace}+if^{dac}if^{bce}+if^{abc}if^{dce}\right)t^e\,.$$

$$(t^a)^i{}_j (t^a)^k{}_l = T_F \left[\delta^i_l \delta^k_j - \frac{1}{N_c} \delta^i_j \delta^k_l \right] \, ,$$

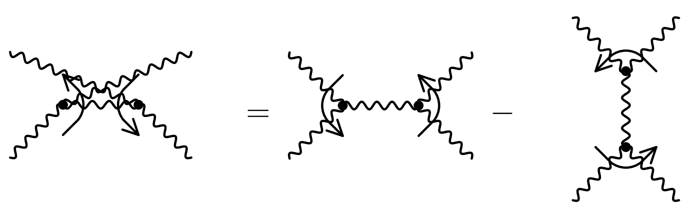


$$f^{abc} \;=\; -\frac{i}{\sqrt{2}} \Big(\mathrm{Tr}\big(T^a T^b T^c\big) - \mathrm{Tr}\big(T^a T^c T^b\big) \Big),$$

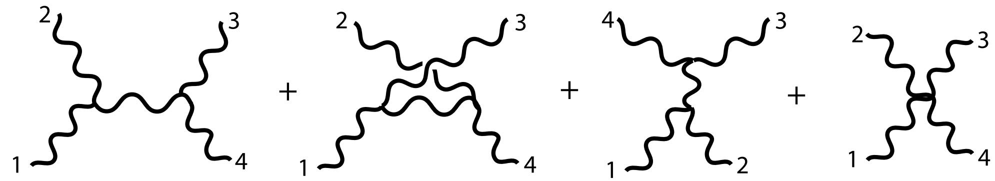


Jacobi identity

$$[t^a, [t^b, t^d]] + [t^b, [t^d, t^a]] + [t^d, [t^a, t^b]] = 0$$
$$= \left(i f^{bdc} i f^{ace} + i f^{dac} i f^{bce} + i f^{abc} i f^{dce} \right) t^e \, .$$



Feynman diagrams for $gg \to gg$.



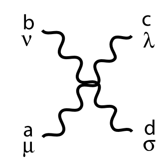
Other QCD amplitudes can also be reduced to color-ordered structures. Another case that will be important for us is the 4-gluon amplitude shown in Fig. 5. This amplitude can be written in the form

$$\begin{aligned} i\mathcal{M} = i\mathbf{M}(1234)\cdot \mathrm{tr}[T^aT^bT^cT^d] + i\mathbf{M}(1243)\cdot \mathrm{tr}[T^aT^bT^dT^c] \\ +i\mathbf{M}(1324)\cdot \mathrm{tr}[T^aT^cT^bT^d] + i\mathbf{M}(1342)\cdot \mathrm{tr}[T^aT^cT^dT^b] \\ +i\mathbf{M}(1423)\cdot \mathrm{tr}[T^aT^dT^bT^c] + i\mathbf{M}(1432)\cdot \mathrm{tr}[T^aT^dT^cT^b]\ . \end{aligned} \tag{50}$$

To write the four diagrams in Fig. 5 as a sum of color structures, we need to convert color factors in the 3- and 4-gluon vertices into products of T^a matrices. For the 3-gluon vertex, this is done through (46), or, by the use of (47), through

$$-gf^{abc} = \frac{ig}{\sqrt{2}}\mathrm{tr}[T^aT^bT^c - T^aT^cT^b]\ . \tag{51}$$

For the 4-gluon vertex, we need to apply this decomposition twice. The textbook form of the 4-gluon vertex is shown in Fig. 6. Each term can be manipulated as follows



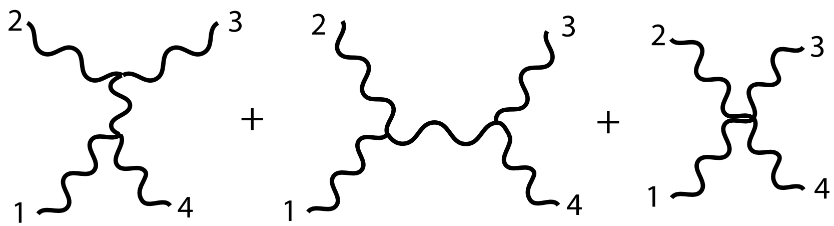
$$= -ig^2\,[\,f^{abe}f^{cde}(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\lambda}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\sigma}g^{\mu\lambda}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma})]$$

$$\begin{aligned} -ig^2f^{abe}f^{cde} &= i\frac{g^2}{2}\mathrm{tr}([T^a,T^b][T^c,T^d]) \\ &= i\frac{g^2}{2}\mathrm{tr}(T^aT^bT^cT^d - T^aT^bT^dT^c - T^bT^aT^cT^d + T^bT^aT^dT^c) \end{aligned} \tag{52}$$

The full 4-gluon vertex can then be rearranged into

$$i\frac{g^2}{2}\mathrm{tr}(T^aT^bT^cT^d)[2g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\sigma}g^{\nu\lambda}] \tag{53}$$

plus 5 more terms corresponding to the other 5 color structures in (50).



As an example, I will compute the color-ordered amplitude $\mathbf{M}(1234)$ that is used to build up the four-gluon amplitude. Of the four Feynman diagrams in Fig. 5, only the three diagrams shown in Fig. 9 contribute to this color-ordered component. Here and in the rest of the lectures, the color-ordered amplitudes in the figures will be ordered clockwise. Using the Feynman rules in Fig. 7, we find

$$
\begin{aligned}
i\mathbf{M}(1234) = (\frac{ig}{\sqrt{2}})^2 \Big[\frac{-i}{s_{14}} & [\epsilon(4)\cdot\epsilon(1)(4-1)^\lambda + \epsilon^\lambda(1)(2\,1+4)\cdot\epsilon(4) + \epsilon^\lambda(4)(-2\,4-1)\cdot\epsilon(1)] \\
& \cdot[\epsilon(2)\cdot\epsilon(3)(2-3)_\lambda + \epsilon_\lambda(3)(2\,3+2)\cdot\epsilon(2) + \epsilon_\lambda(2)(-2\,2-3)\cdot\epsilon(3)] \\
& + \frac{-i}{s_{12}}[\epsilon(1)\cdot\epsilon(2)(1-2)^\lambda + \epsilon^\lambda(2)(2\,2+1)\cdot\epsilon(1) + \epsilon^\lambda(1)(-2\,1-2)\cdot\epsilon(2)] \\
& \cdot[\epsilon(3)\cdot\epsilon(4)(3-4)_\lambda + \epsilon_\lambda(4)(2\,4+3)\cdot\epsilon(3) + \epsilon_\lambda(3)(-2\,3-4)\cdot\epsilon(2)] \\
& + (-i)[2\epsilon(1)\cdot\epsilon(3)\,\epsilon(2)\cdot\epsilon(4) - \epsilon(1)\cdot\epsilon(2)\,\epsilon(3)\cdot\epsilon(4) - \epsilon(1)\cdot\epsilon(4)\,\epsilon(2)\cdot\epsilon(3)] \Big] \, .
\end{aligned}
$$

Feynman rules for colour subamplitudes (for massless fermions)

We introduce the following notation:

- external outgoing fermion of momentum p, helicity $\pm$: $\langle p\pm|$,
- external outgoing antifermion of momentum p, helicity $\pm$: $|p\mp\rangle$,
- external outgoing vector of momentum p, reference k, helicity $\pm$: $\epsilon_\mu^\pm(p,k) = \pm\frac{\langle p\pm|\gamma_\mu|k\pm\rangle}{\sqrt{2}\langle k\mp|p\pm\rangle}$.

The fermion propagator of momentum p in the direction of the fermion arrow is $+\mathrm{i}\frac{\not p}{p^2}$. The vector propagator of momentum p is: $-\frac{\mathrm{i}}{p^2}g_{\mu\nu}$.

$$\Gamma^\mu_{gq\bar{q}} = \mathrm{i}\frac{g_s}{\sqrt{2}}\gamma^\mu \qquad \text{the factor of } \sqrt{2} \text{ is due to } T_R = 1. \tag{1.48}$$

$$\Gamma_{\alpha\beta\gamma}(p,q,r) = \mathrm{i}\frac{g_s}{\sqrt{2}}V_{\alpha\beta\gamma}(p,q,r) \qquad \text{all incoming momenta} \tag{1.49}$$

$$\Gamma_{\alpha\beta\gamma\delta} = \mathrm{i}\frac{g_s^2}{2}\left(2g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\beta}g_{\gamma\delta}\right) \tag{1.50}$$

the case with all positive helicities. Choose the gluon polarization vectors so that the

same reference vector r is used in every case,

$$\epsilon^\mu(j) = \frac{1}{\sqrt{2}} \frac{\langle r\gamma^\mu j]}{\langle rj\rangle} \ . \tag{68}$$

Then, for all i, j,

$$\epsilon(i) \cdot \epsilon(j) \sim \langle rr\rangle [ji] = 0 \ . \tag{69}$$

By inspection of (56), every term contains at least one factor of $\epsilon(i) \cdot \epsilon(j)$. Thus, the entire expression vanishes.

This argument is easily extended to the case with one negative helicity gluon. The amplitude (56) is cyclically symmetric, so we can chose the gluon 1 to have negative helicity without loss of generality. Then let

$$\epsilon^\mu(1) = -\frac{1}{\sqrt{2}} \frac{[2\gamma^\mu 1\rangle}{[21]} \ , \qquad \epsilon^\mu(j) = \frac{1}{\sqrt{2}} \frac{\langle 1\gamma^\mu j]}{\langle 1j\rangle} \ , \tag{70}$$

for $j = 2, 3, 4$. Again, $\epsilon(i) \cdot \epsilon(j) = 0$ for all i, j, and so the complete amplitude vanishes.

It is not difficult to see that these arguments carry over directly to the n-gluon color-ordered amplitudes for any value of n. The tree amplitudes with all positive helicities, or with one negative helicity and all of the rest positive, vanish. The *maximally helicity violating* amplitudes are those with two negative and the rest positive helicities.

For the 4-gluon amplitude, all that remains is to compute the color-ordered amplitude in the case with two negative helicities. By the cyclic invariance of $\mathbf{M}$, there are only two cases, that in which the two negative helicities are adjacent and that in which they are opposite. As an example of the first case, we can analyze $i\mathbf{M}(1_-2_-3_+4_+)$. Choose the polarization vectors to be

$$\begin{aligned} \epsilon^\mu(1) &= -\frac{1}{\sqrt{2}}\frac{[4\gamma^\mu 1\rangle}{[41]} \qquad & \epsilon^\mu(2) &= -\frac{1}{\sqrt{2}}\frac{[4\gamma^\mu 2\rangle}{[42]} \\ \epsilon^\mu(3) &= \frac{1}{\sqrt{2}}\frac{\langle 1\gamma^\mu 3]}{\langle 13\rangle} \qquad & \epsilon^\mu(4) &= \frac{1}{\sqrt{2}}\frac{\langle 1\gamma^\mu 4]}{\langle 14\rangle} \end{aligned} \tag{71}$$

With this choice, all scalar products of ϵ's are zero except for

$$\epsilon(2)\cdot\epsilon(3) = -\frac{1}{2}\frac{2[43]\langle 12\rangle}{[42]\langle 13\rangle} = -\frac{\langle 12\rangle[43]}{\langle 13\rangle[42]} \tag{72}$$

Looking back at (56), we see that the first and third lines are zero. In the second line, the only nonzero term is the one that involves $\epsilon(2)\cdot\epsilon(3)$ and no other dot product of ϵ's. Thus,

$$\begin{aligned} i\mathbf{M} &= (\frac{ig^2}{2})\frac{-i}{s_{34}}(-4)\epsilon(2)\cdot\epsilon(3)\,2\cdot\epsilon(1)\,3\cdot\epsilon(4) \\ &= -ig^2\frac{\langle 12\rangle^2[34]}{\langle 34\rangle[41]\langle 41\rangle} \end{aligned} \tag{73}$$

Multiplying top and bottom by $\langle 12\rangle$ and rearranging terms in the denominator to cancel out the square bracket factors, we find

$$i\mathbf{M}(1_-2_-3_+4_+) = ig^2\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\ . \tag{74}$$

Similarly, to evaluate $i\mathbf{M}(1_-2_+3_-4_+)$, choose the polarization vectors to be

$$\begin{aligned} \epsilon^\mu(1) &= -\frac{1}{\sqrt{2}}\frac{[4\gamma^\mu 1\rangle}{[41]} & \epsilon^\mu(2) &= \frac{1}{\sqrt{2}}\frac{\langle 1\gamma^\mu 2]}{\langle 12\rangle} \\ \epsilon^\mu(3) &= -\frac{1}{\sqrt{2}}\frac{[4\gamma^\mu 3\rangle}{[43]} & \epsilon^\mu(4) &= \frac{1}{\sqrt{2}}\frac{\langle 1\gamma^\mu 4]}{\langle 14\rangle} \end{aligned} \tag{75}$$

With this choice, all scalar products of ϵ's are zero except for

$$\epsilon(2)\cdot\epsilon(3) = -\frac{\langle 13\rangle[42]}{\langle 12\rangle[43]} \tag{76}$$

Again, only the term that involves $\epsilon(2)\cdot\epsilon(3)$ and no other dot product of ϵ's is nonzero. The value of that term is again given by the first line of (73), which, in this case, evaluates to

$$i\mathbf{M} = -ig^2\frac{\langle 13\rangle^2[42]^2}{\langle 34\rangle[41]\langle 41\rangle[43]}\ . \tag{77}$$

Multiplying top and bottom by $\langle 13\rangle^2$ and rearranging terms in the denominator to cancel out the square brackets, we find

$$i\mathbf{M}(1_-2_+3_-4_+) = ig^2\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\ . \tag{78}$$

The form of (74) and (78) strongly suggests that the general form of an n-gluon MHV amplitude is

$$i\mathbf{M}(g_+(1)\cdots g_-(i)\cdots g_-(j)\cdots g_+(n)) = ig^{n-2}\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle} \ . \tag{79}$$

The corresponding formula, exchanging positive and negative helicities, is

$$i\mathbf{M}(g_-(1)\cdots g_+(i)\cdots g_+(j)\cdots g_-(n)) = (-1)^n ig^{n-2}\frac{[ij]^4}{[12][23]\cdots[(n-1)n][n1]} \ . \tag{80}$$

discovered in 1986 by Parke and Taylor

Photondecoupling equation

the lagrangian for SU(N) Yang–Mills theory

$$\mathcal{L} = \mathrm{Tr}\Big(-\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - i\sqrt{2}g\,\partial^\mu A^\nu A_\nu A_\mu + \tfrac{1}{4}g^2 A^\mu A^\nu A_\mu A_\nu\Big)\;, \qquad (81.1)$$

where $A_\mu(x)$ is a traceless hermitian $N \times N$ matrix. For quantum chromodynamics, $N = 3$, but we will leave N unspecified in our calculations. In section 80, we worked out the *color-ordered Feynman rules* for a scalar matrix field; the same technology applies here as well. In particular, we draw each tree diagram in planar fashion (that is, with no crossed lines). Then the cyclic, counterclockwise ordering $i_1 \ldots i_n$ of the external lines fixes the color factor as $\mathrm{Tr}(T^{a_{i_1}} \ldots T^{a_{i_n}})$, where the generator matrices are normalized via $\mathrm{Tr}(T^a T^b) = \delta^{ab}$. The tree-level n-gluon scattering amplitude is then written as

$$\mathcal{T} = g^{n-2} \sum_{\substack{\text{noncyclic}\\ \text{perms}}} \mathrm{Tr}(T^{a_1} \ldots T^{a_n}) A(1, \ldots, n)\;, \qquad (81.2)$$

where we have pulled out the coupling constant dependence, and $A(1, \ldots, n)$ is a *partial amplitude* that we compute with the color-ordered Feynman rules. The partial amplitudes are cyclically symmetric,

$$A(2, \ldots, n, 1) = A(1, 2, \ldots, n)\;. \qquad (81.3)$$

The sum in eq. (81.2) is over all *noncyclic* permutations of $1 \ldots n$, which is equivalent to a sum over *all* permutations of $2 \ldots n$.

completeness relation

$$(T^a)_i{}^j (T^a)_k{}^l = \delta_i{}^l \delta_k{}^j - \tfrac{1}{N} \delta_i{}^j \delta_k{}^l \,. \tag{81.30}$$

However, recall that the Yang–Mills field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}[A_\mu,A_\nu] \,. \tag{81.31}$$

If we allow a generator matrix proportional to the identity, which corresponds to a gauge group of U(N) rather than SU(N), then this extra U(1) generator commutes with every other generator. Thus the U(1) field does not appear in the commutator term in eq. (81.31). Since it is this commutator term that is responsible for the interaction of the gluons, the U(1) field is a free field. Therefore, any scattering amplitude involving the associated particle (which we will call the *fictitious photon*) must be zero. Thus, if we write a scattering amplitude in the form of eq. (81.2), and replace one of the T^a's with the identity matrix, the result must be zero.

This *decoupling of the fictitious photon* allows us to use the much simpler completeness relation

$$(T^a)_i{}^j (T^a)_k{}^l = \delta_i{}^l \delta_k{}^j \tag{81.32}$$

in place of eq. (81.30). There is no need to subtract the U(1) generator from the sum over the generators, as we did in eq. (81.30), because the terms involving it vanish anyway.

we want to extend our gauge group from $SU(N)$ to $U(N) = SU(N) \times U(1)$. We need to add the extra $U(1)$ generator. This would normally correspond to a diffrent theory, but here we have a special case, since this $U(1)$ generator corresponds to the unit matrix it will commute with all the generators of $SU(N)$. This means that we can't have a scattering amplitude involving this particle, because the three and four point vertices are proportional to commutators of the corresponding generators. Therefor this transsion can't alter our final result.

It is easy to see that any tree diagram for n-gluon scattering can be reduced to a sum of "single trace" terms. This observation leads to the *color decomposition* of the the n-gluon tree amplitude,[6]

$$\mathcal{A}_n^{\rm tree}\left(\{k_i,\lambda_i,a_i\}\right) = g^{n-2}\sum_{\sigma\in S_n/Z_n} {\rm Tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}\right)\ A_n^{\rm tree}(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n})). \tag{4}$$

Here g is the gauge coupling ($\frac{g^2}{4\pi}=\alpha_s$), k_i,λ_i are the gluon momenta and helicities, and $A_n^{\rm tree}(1^{\lambda_1},\ldots,n^{\lambda_n})$ are the *partial amplitudes*, which contain all the kinematic information. S_n is the set of all permutations of n objects, while Z_n is the subset of cyclic permutations, which preserves the trace; one sums over the set S_n/Z_n in order to sweep out all distinct cyclic orderings in the trace. The real work is still to come, in calculating the independent partial amplitudes $A_n^{\rm tree}$. However, the partial amplitudes are simpler than the full amplitude because they are *color-ordered*: they only receive contributions from diagrams with a particular cyclic ordering of the gluons. Because of this, the singularities of the partial amplitudes, poles and (in the loop case) cuts, can only occur in a limited set of momentum channels, those made out of sums of cyclically adjacent momenta. For example, the five-point partial amplitudes $A_5^{\rm tree}(1^{\lambda_1},2^{\lambda_2},3^{\lambda_3},4^{\lambda_4},5^{\lambda_5})$ can only have poles in s_{12}, s_{23}, s_{34}, s_{45}, and s_{51}, and not in s_{13}, s_{24}, s_{35}, s_{41}, or s_{52}, where $s_{ij}\equiv(k_i+k_j)^2$.

tree color decomposition, Eq. 4, is equally valid for gauge group $U(N_c)$ as $SU(N_c)$, but any amplitude containing the extra $U(1)$ photon must vanish. Hence if we substitute the $U(1)$ generator — the identity matrix — into the right-hand-side of Eq. 4, and collect the terms with the same remaining color structure, that linear combination of partial amplitudes must vanish. We get

$$\begin{aligned} 0 \ =\ & A_n^{\rm tree}(1,2,3,\ldots,n)+A_n^{\rm tree}(2,1,3,\ldots,n)+A_n^{\rm tree}(2,3,1,\ldots,n)\\ & +\cdots+A_n^{\rm tree}(2,3,\ldots,1,n), \end{aligned}\tag{8}$$

often called a "photon decoupling equation"[7] or "dual Ward identity"[3]

The decoupling of the fictitious photon is useful in another way. Let us apply it to the case of $n = 4$, and set $T^{a_4} \propto I$ in eq. (81.2). Then we have

$$
\begin{aligned}
A_4^{tree} = g^2 &\left[Tr(T^1T^2T^3T^4)A(1,2,3,4) + Tr(T^1T^2T^4T^3)A(1,2,4,3)\right] \\
+g^2 &\left[Tr(T^1T^3T^2T^4)A(1,3,2,4) + Tr(T^1T^4T^2T^3)A(1,4,2,3)\right] \qquad (1) \\
+g^2 &\left[Tr(T^1T^3T^4T^2)A(1,3,4,2) + Tr(T^1T^4T^3T^2)A(1,4,3,2)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= g^2 A(1,2,3,4)\left[Tr(T^1T^2T^3T^4) + Tr(T^1T^4T^3T^2)\right] \\
&+g^2 A(1,4,2,3)\left[Tr(T^1T^4T^2T^3) + Tr(T^1T^3T^2T^4)\right] \qquad (2) \\
&+g^2 A(1,3,4,2)\left[Tr(T^1T^3T^4T^2) + Tr(T^1T^2T^4T^3)\right]
\end{aligned}
$$

$$
\begin{aligned}
0 &= g^2 A(1,2,3,4)\left[Tr(T^1T^2T^3) + Tr(T^1T^3T^2)\right] \\
&+g^2 A(1,4,2,3)\left[Tr(T^1T^2T^3) + Tr(T^1T^3T^2)\right] \qquad (4) \\
&+g^2 A(1,3,4,2)\left[Tr(T^1T^3T^2) + Tr(T^1T^2T^3)\right]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow 0 = g^2 &\left(A(1,2,3,4) + A(1,4,2,3) + A(1,3,4,2)\right) \\
&\left[Tr(T^1T^2T^3) + Tr(T^1T^3T^2)\right] \qquad (5)
\end{aligned}
$$

the 2nd factor can be written as

$$
Tr(T^aT^bT^c + T^aT^cT^b) = Tr(T^a\left\{T^b, T^c\right\}) = d^{abc} \qquad (6)
$$

where d^{abc} are also structure constants of $SU(3)$

$$
A(1,2,3,4) = -A(1,2,4,3) - A(1,4,2,3)\;.
$$