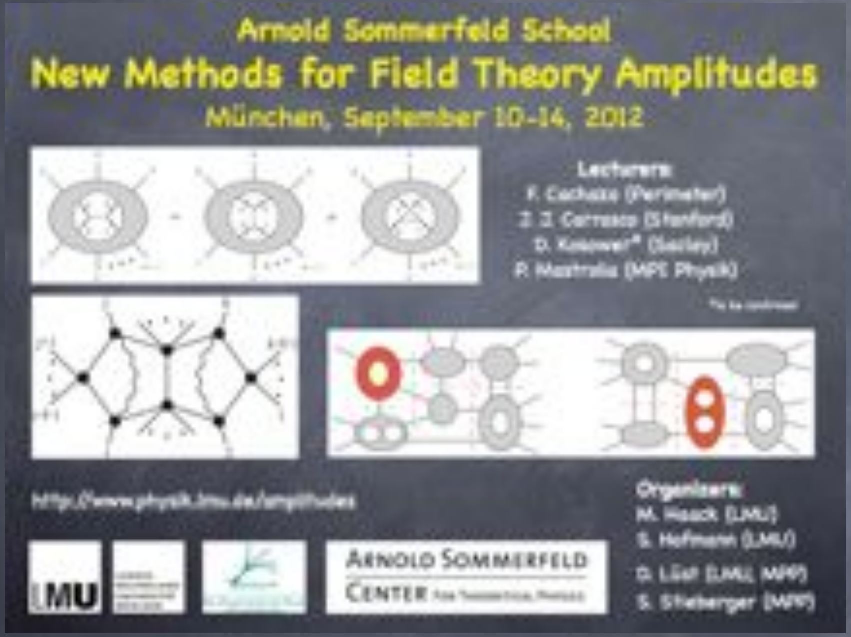
Lecture 4 slides
ASC School 2012



Arnold Sommerfeld School

New Methods for Field Theory Amplitudes

München, September 10-14, 2012

Lecturers:
F. Cachazo (Perimeter)
J. J. Carrasco (Stanford)
D. Kosower* (Saclay)
P. Mastrolia (MPI Physik)

*to be confirmed

http://www.physik.lmu.de/amplitudes

Organizers:
M. Haack (LMU)
S. Hofmann (LMU)
D. Lüst (LMU, MPP)
S. Stieberger (MPP)

John Joseph M. Carrasco
Stanford Institute for Theoretical Physics

Some References re: tree duality satisfying numerators:

Since **BCJ '08** there have been many interesting ways of writing down tree-level color-kinematic satisfying numerators

Rearranging the Lagrangian:
(also a field theory proof at tree level c-k => double copy)
Bern, Dennen, Huang, Kiermaier '10

Teasing c-k numerators out of KLT:
Kiermaier '10
Bjerrum-Bohr, Damgaard, Sondegaard, Vanhove '10

String-insight & pure spinors:
Tye, Zhang'11
Mafra, Schlotterer, Stieberger '11

Self-dual understanding -> MHV:
Montiero, O'Connell '11

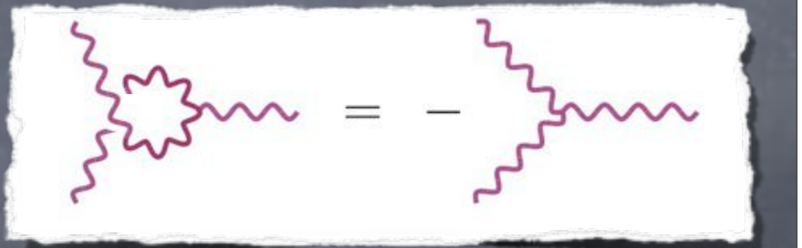
Constructing effective field theories:
Bjerrum-Bohr, Damgaard, Monteiro, O'Connell '12

Applying loop-methods:
Broedel, JJMC '11

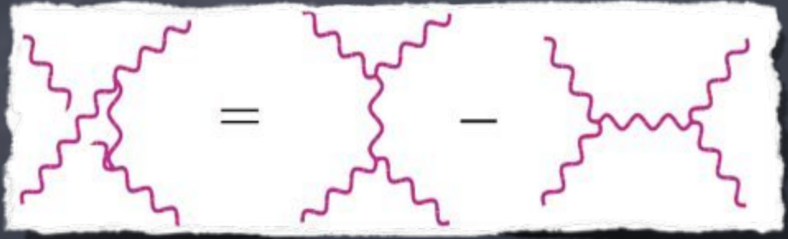
BCJ '08

Graph rep at tree level

- Antisymmetry of kinematic numerators makes manifest (n-2)! basis relations (Kleiss Kuijf relations) between color ordered amplitudes



- After full color-kinematic duality imposed (kinematic Jacobi), makes manifest a (n-3)! basis



Bjerrum-Bohr, Damgaard, Vanhove '09; Stieberger '09

STRINGS

Monodromy relations in open string leads to string generalization of (n-2)! Kleiss-Kuijf and (n-3)! relations and thus string proof of field theory relations as real and imaginary parts with $\alpha' \to 0$

$$A(1,2,\ldots,N) \; + e^{i\pi s_{12}} \; A(2,1,3,\ldots,N-1,N) + e^{i\pi(s_{12}+s_{13})} \; A(2,3,1,\ldots,N-1,N)$$
$$+ \ldots + e^{i\pi(s_{12}+s_{13}+\ldots+s_{1N-1})} \; A(2,3,\ldots,N-1,1,N) = 0$$

Real part yields Kleiss-Kuijf (n-2)!:

$$A_{YM}(1,2,\ldots,N) + A_{YM}(2,1,3,\ldots,N-1,N) + \ldots + A_{YM}(2,3,\ldots,N-1,1,N) = 0$$

Imaginary part yields (n-3)! :

$$s_{12} \; A_{YM}(2,1,3,\ldots,N-1,N) + \ldots + (s_{12}+s_{13}+\ldots+s_{1N-1}) \; A(2,3,\ldots,N-1,1,N) = 0$$

QFT

Feng, (R) Huang, Jia '10; Jia, (R) Huang, Liu '10; Cachazo '12

Bringing power of BCFW to bear, direct all multiplicity field theory proofs of (n-3)! relations.

Mafra, Schlotterer, Stieberger '11

Understanding the string roots of (n-2)! and (n-3)! relations led to **complete pure spinor n-point open-disk amplitude** in terms of color ordered gauge theory amplitudes!

$$\boxed{\mathcal{A}(1,2,\ldots,N;\alpha') \;=\; \sum_{\pi\in S_{N-3}} \mathcal{A}^{\rm YM}\big(1,2_\pi,\ldots,(N-2)_\pi,N-1,N\big)\,F^\pi(\alpha')}$$

- decomposes into $(N-3)!$ field theory subamplitudes $\mathcal{A}^{\rm YM}_{\pi\in S_{N-3}}$
- string effects (α' dependence) from generalized Euler integrals $F^\pi(\alpha')$

BDPR-KLT field theory expressions:

Kawai, Lewellen, Tye

Bern, Dixon, Perelstein, Rozowsky ('97)

Gravity tree amplitudes:

$$\mathsf{M}_n^{\text{tree}}(1,\ldots,n-1,n) = i(-1)^{n+1} \sum_{perms(2,\ldots,n-2)} \Big[A^{\text{tree}}(1,\ldots,n-1,n)$$

$$\times \sum_{perms(i,l)} f(i)\bar{f}(l)\tilde{A}_n^{\text{tree}}(i_1,\ldots,i_{(n/2-1)},1,n-1,l_1,\ldots,l_{n/2-2},n)\Big]$$

Color-ordered gauge tree amplitudes

$$\mathbf{i} = \text{perm}(\{2,\ldots,n/2\})$$

$$l = \text{perm}(\{n/2+1,\ldots,n-2\})$$

$$\mathbf{f}(i_1,\ldots,i_j) \quad = \quad s_{1,i_j} \prod_{m=1}^{j-1} \left(s_{1,i_m} + \sum_{k=m+1}^{j} g(i_m,i_k) \right),$$

$$\bar{f}(l_1,\ldots,l_{j'}) \quad = \quad s_{l_1,n-1} \prod_{m=2}^{j'} \left(s_{l_m,n-1} + \sum_{k=1}^{m-1} g(l_k,l_m) \right)$$

$$\mathbf{g}(i,j) = \left\{ \begin{array}{ll} s_{i,j} & \text{if } i > j \\ 0 & \text{else} \end{array} \right\} \qquad \mathbf{s}_{a,b} = (k_a+k_b)^2$$

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Color-ordered gauge tree amplitudes

$$\mathbf{i} = \mathrm{perm}(\{2,\ldots,n/2\})$$
$$l = \mathrm{perm}(\{n/2+1,\ldots,n-2\})$$

New (n-3)! amplitude relations allowed re-expression of field theory KLT in terms of different "basis" amplitudes: Left-right symmetric, etc.

BCJ '08; Bjerrum-Bohr, Damgaard, Feng, Søndergaard '10;

These relations allowed proofs of KLT for gravity and gauge amplitudes in field theory:

Bjerrum-Bohr, Damgaard, Feng, Søndergaard '10; Du, Feng, Fu '11;

Generalized (monodromy) relations allowed rewriting of String Theory KLT in closed form: "momentum-kernel"

Bjerrum-Bohr, Damgaard, Søndergaard, Vanhove '10

BDPR-KLT field theory expressions:

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Color-ordered gauge tree amplitudes

$$\mathbf{i} = \mathrm{perm}(\{2, \ldots, n/2\})$$
$$l = \mathrm{perm}(\{n/2+1, \ldots, n-2\})$$

Of course a natural way to get different "KLT" expressions is to start with

$$-iM_n^{\text{tree}} = \sum_{\mathcal{G}\in\text{cubic}} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

and express in "Amplitude encoded" representations

Bargheer, He, and McLoughlin '12

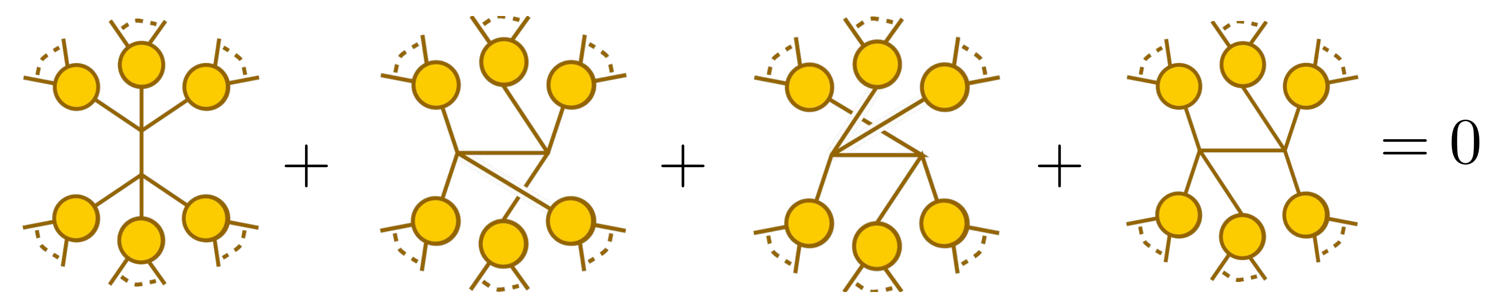
Duality for BLG Theory

Bagger,Lambert,Gustavsson (BLG)

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d$$

D=3 Chern-Simons gauge theory

Generalized Color-Kinematics identity:

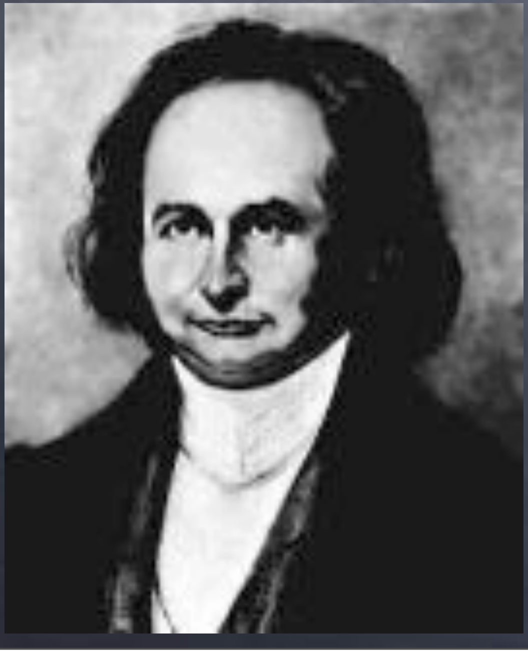


Verified at 4 and 6 point. Double copy gives correct N=16 SUGRA in 3D of Marcus and Schwarz.

!! Very cool result !!

Tree stuff is all (semi)-classical

- The world is QUANTUM - wouldn't it be great to generalize to loop-order corrections?



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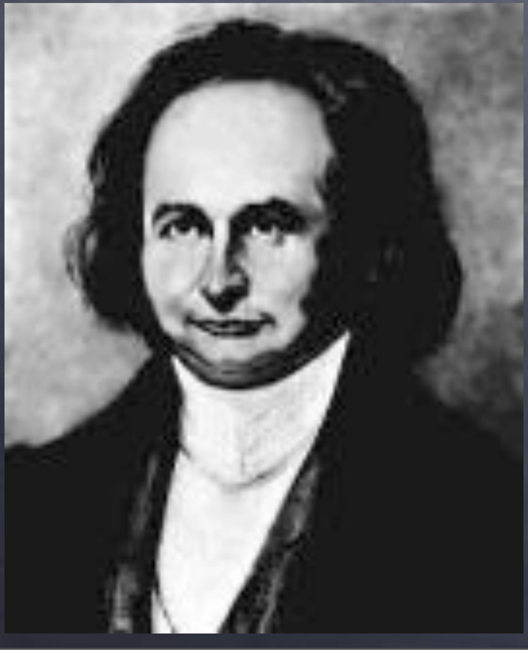


"One should always generalize." - C. Jacobi

What's the right generalization?

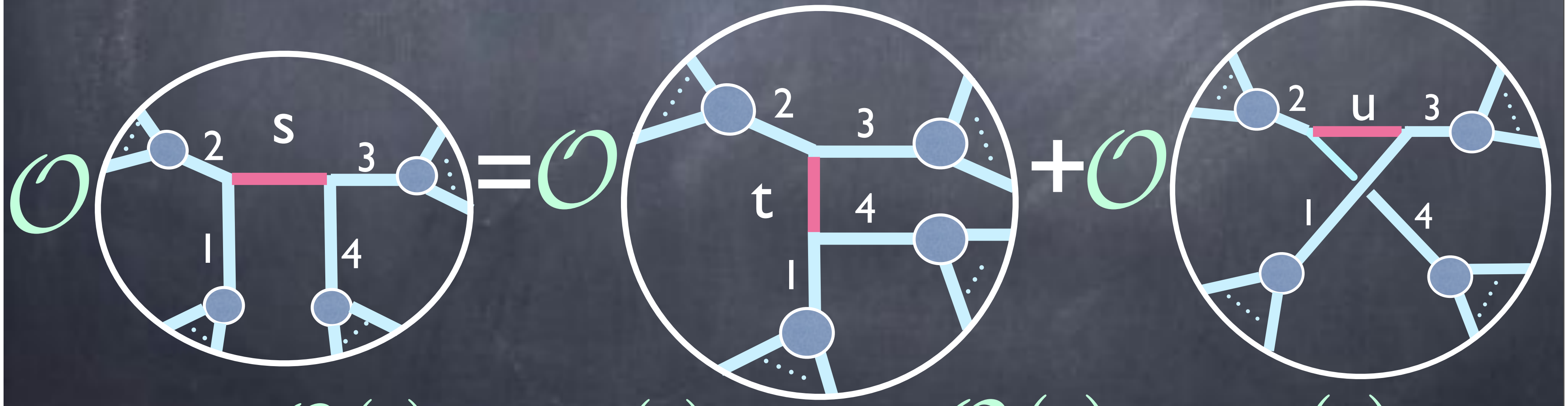
$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})c(\mathcal{G})}{D(\mathcal{G})}$$

Hypothesize duality holds unchanged to all loops!

Representation freedom:

$$n(\mathcal{G}) \to n(\mathcal{G}) + \Delta(\mathcal{G}), \quad \sum_{\mathcal{G}\in\text{cubic}} \left(\frac{c(\mathcal{G})\Delta(\mathcal{G})}{D(\mathcal{G})}\right) = 0$$

Conjecture there is always a choice of Δ such that C-K rep exists.



$$\mathcal{O}(.) = n(.) \longleftrightarrow \mathcal{O}(.) = c(.)$$

If conjectured duality can be imposed for:

Gauge:

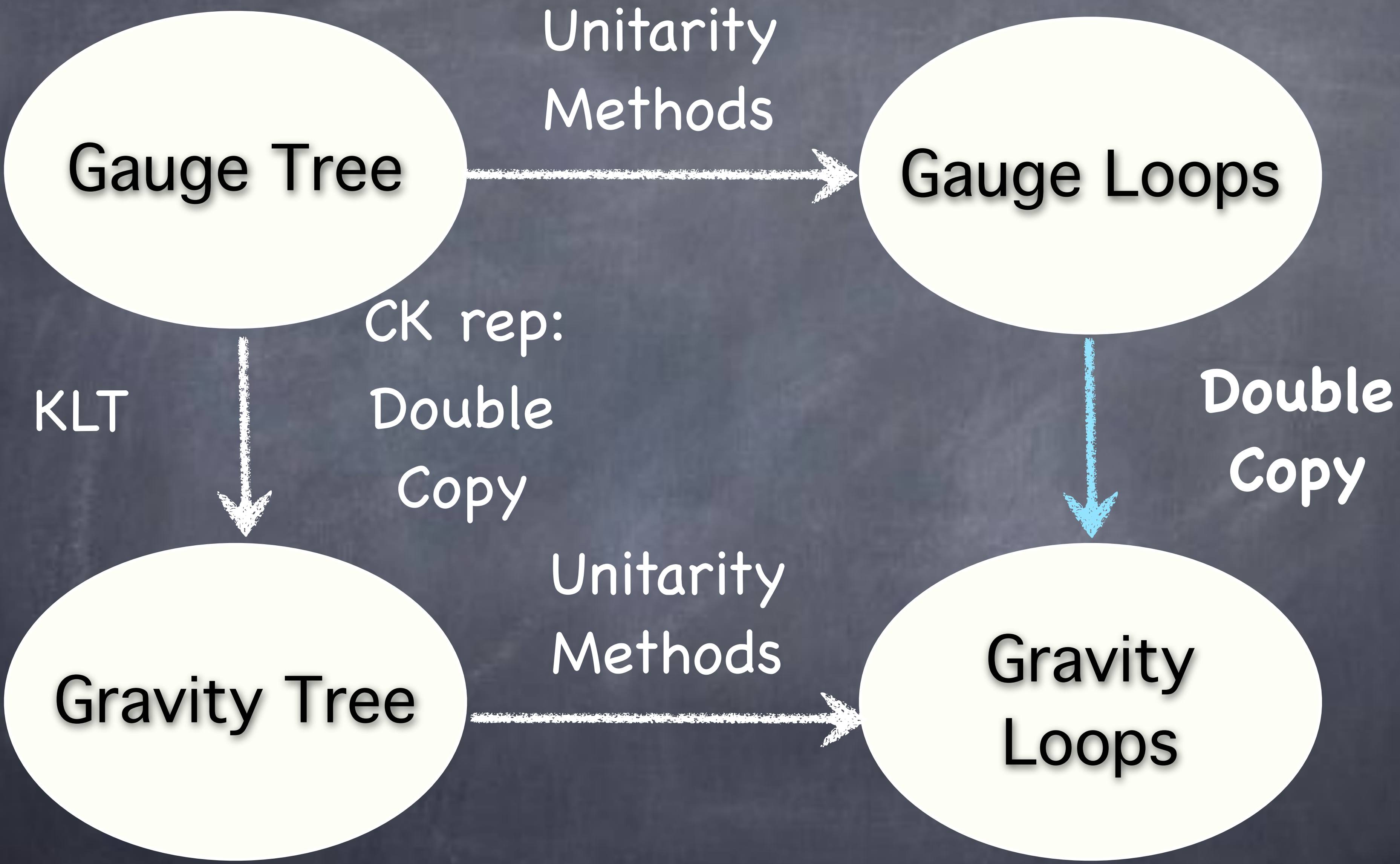
$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})c(\mathcal{G})}{D(\mathcal{G})}$$

then, through unitarity & tree-level expressions:

Gravity:

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}^{\text{loop}} = \sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

What we always wanted out of "loop level" relations!



Gauge Tree

Unitarity Methods

Gauge Loops

KLT

CK rep: Double Copy

Double Copy

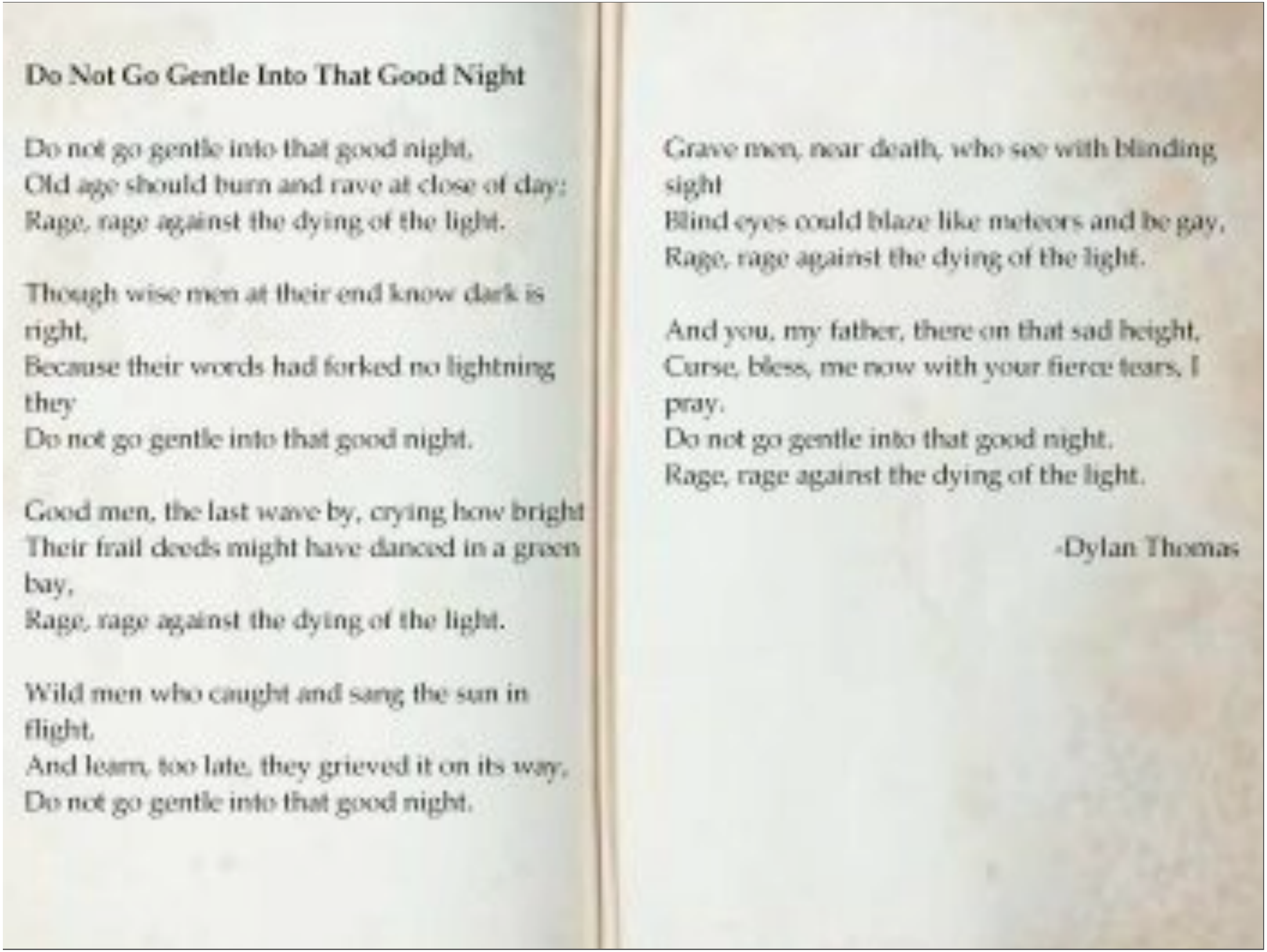
Gravity Tree

Unitarity Methods

Gravity Loops

To appreciate consequences of imposing Jacobi Relations at loop level

Consider a Villanelle



Do Not Go Gentle Into That Good Night

Do not go gentle into that good night,
Old age should burn and rave at close of day;
Rage, rage against the dying of the light.

Though wise men at their end know dark is right,
Because their words had forked no lightning they
Do not go gentle into that good night.

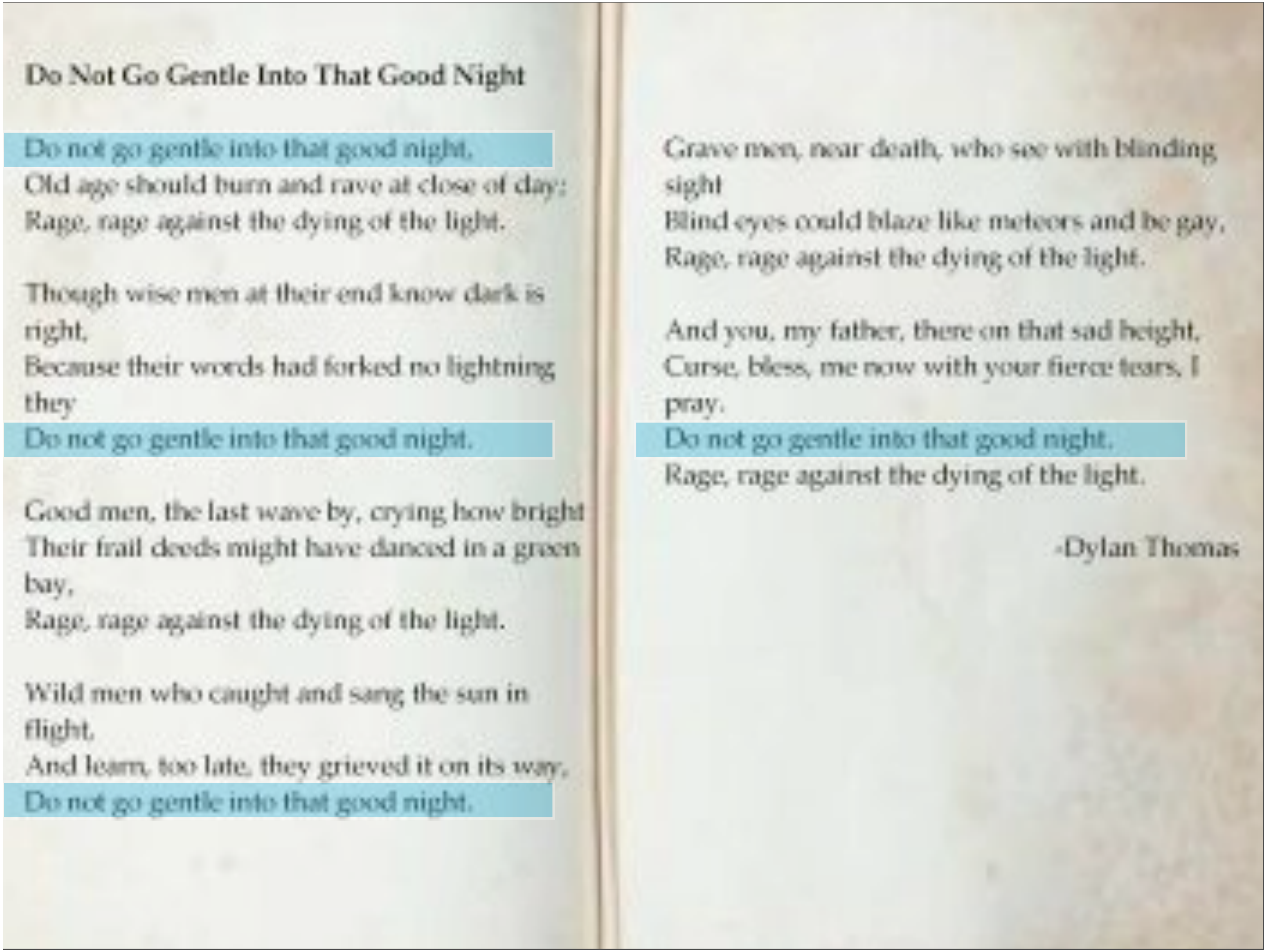
Good men, the last wave by, crying how bright
Their frail deeds might have danced in a green bay,
Rage, rage against the dying of the light.

Wild men who caught and sang the sun in flight,
And learn, too late, they grieved it on its way,
Do not go gentle into that good night.

Grave men, near death, who see with blinding sight
Blind eyes could blaze like meteors and be gay,
Rage, rage against the dying of the light.

And you, my father, there on that sad height,
Curse, bless, me now with your fierce tears, I pray.
Do not go gentle into that good night.
Rage, rage against the dying of the light.

-Dylan Thomas



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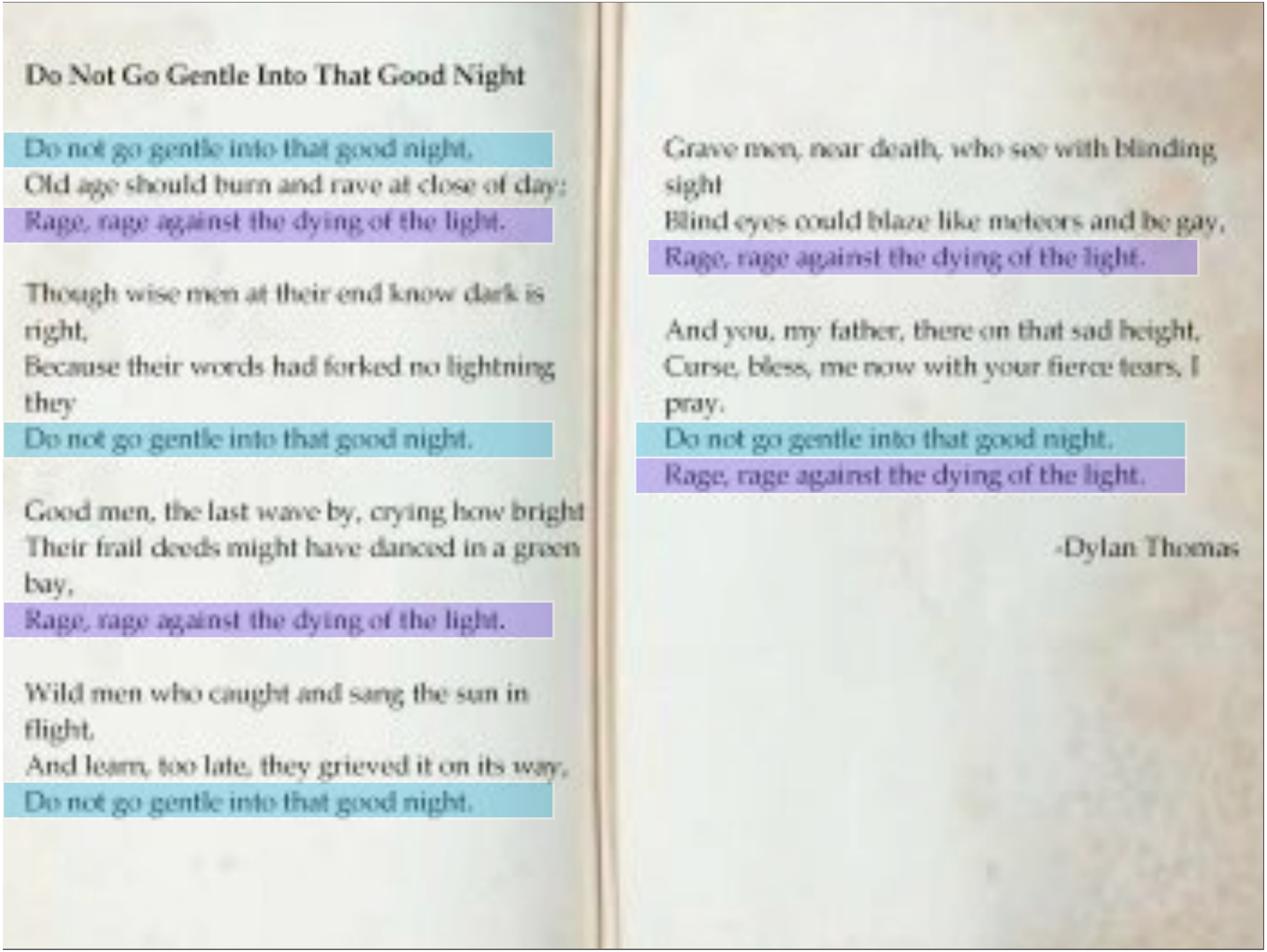

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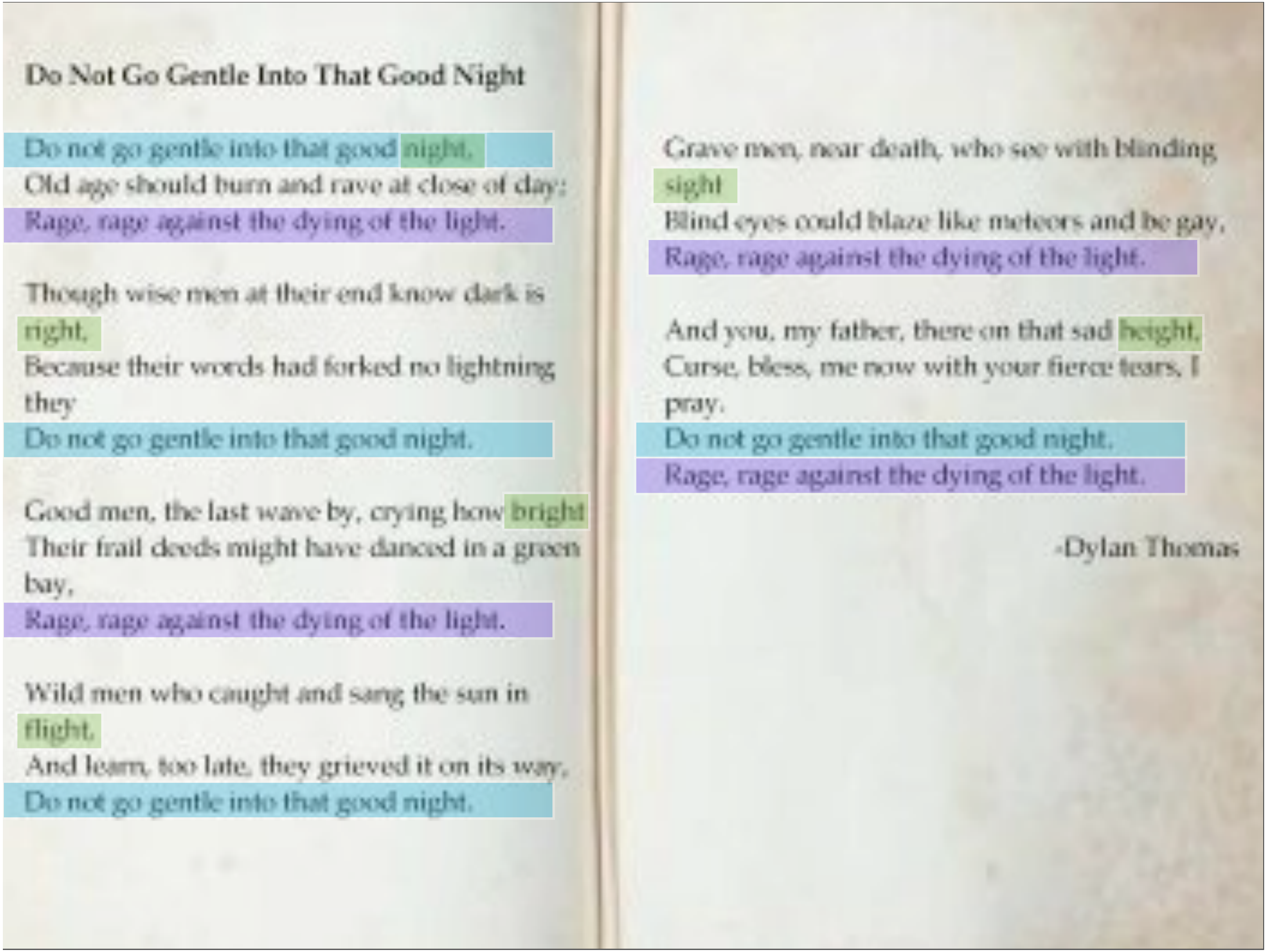

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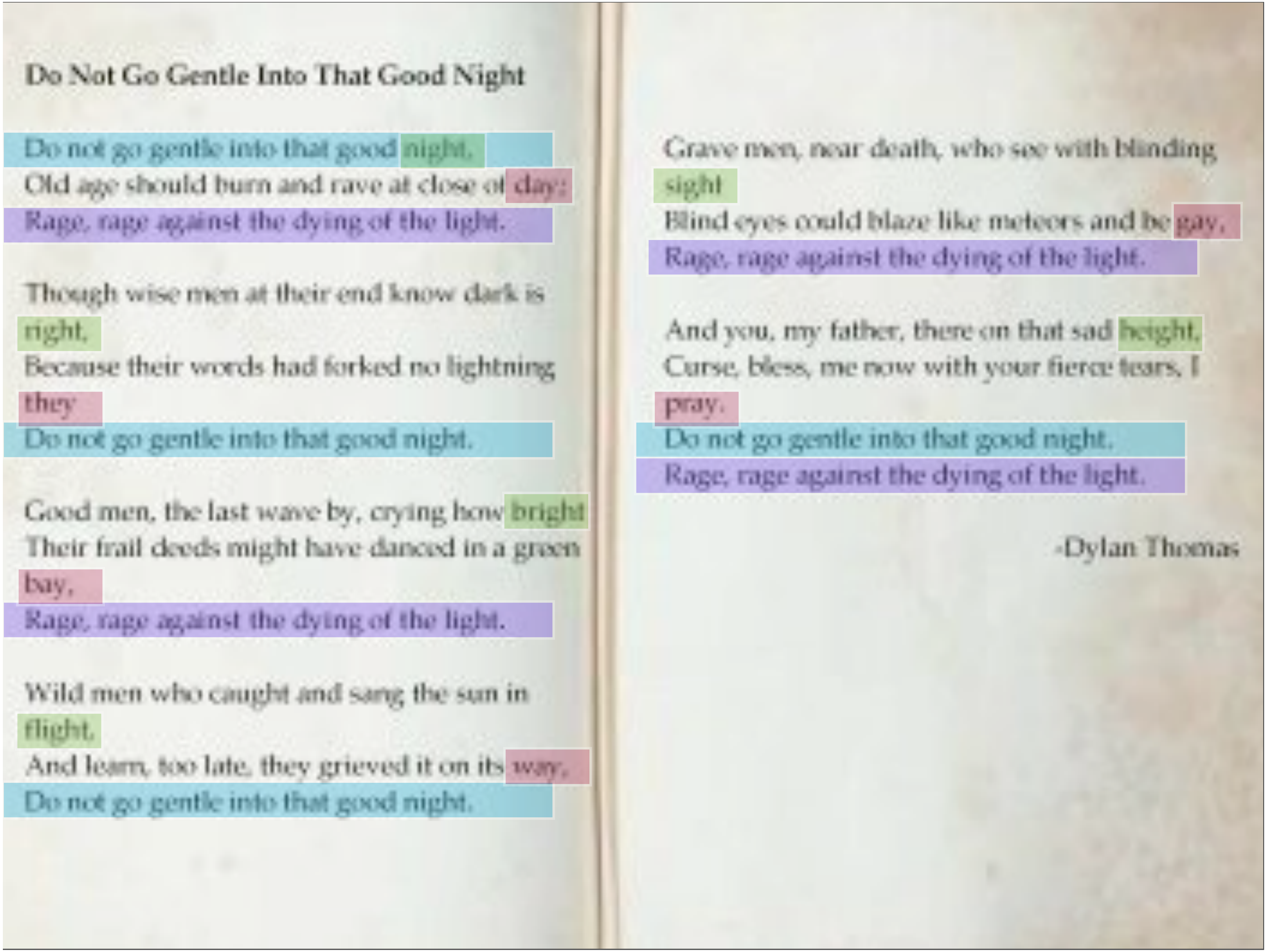

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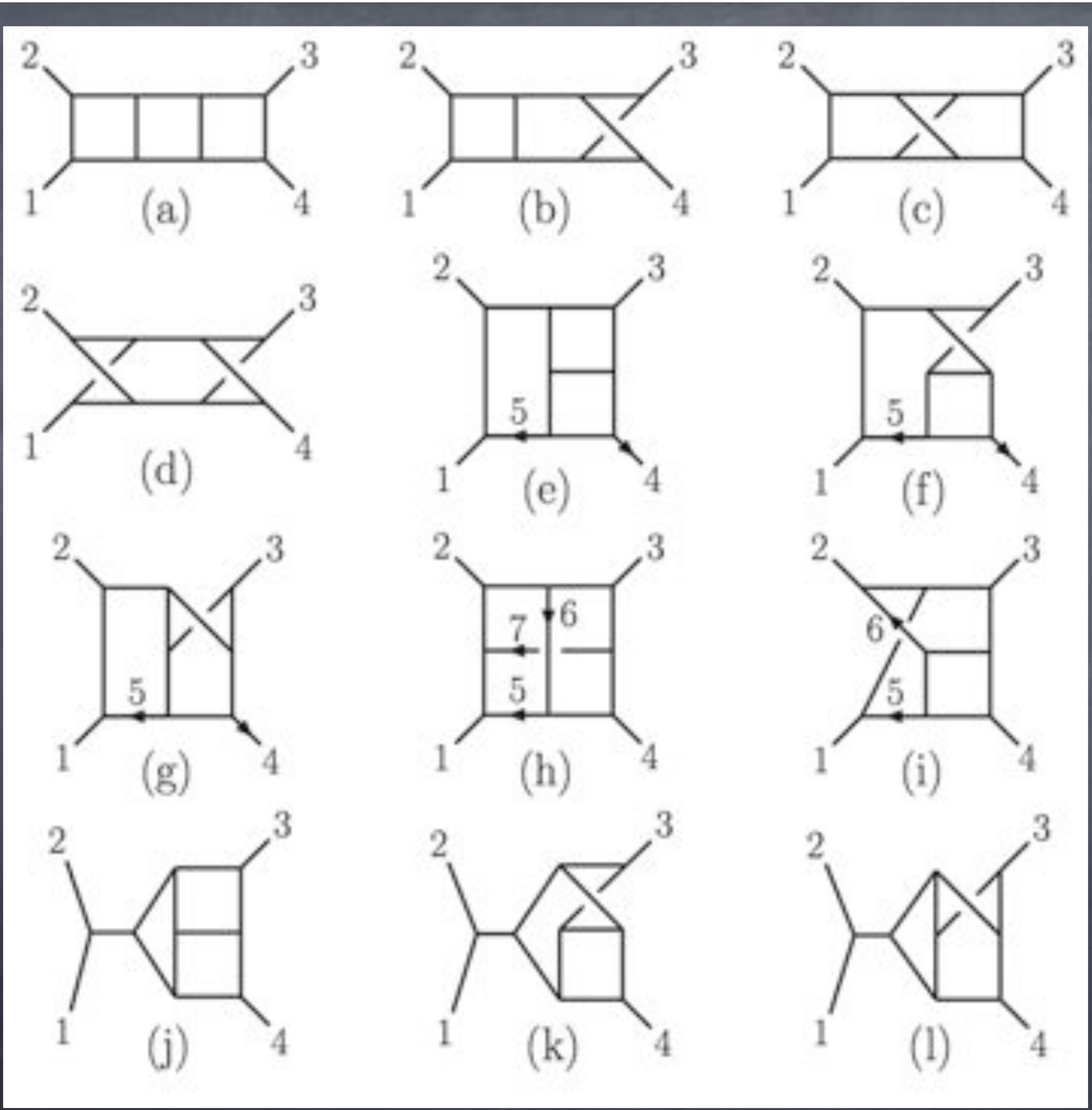
What's going on?

- Minimal information in.
- Relations propagate this information to a full solution.

Consider an Amplitude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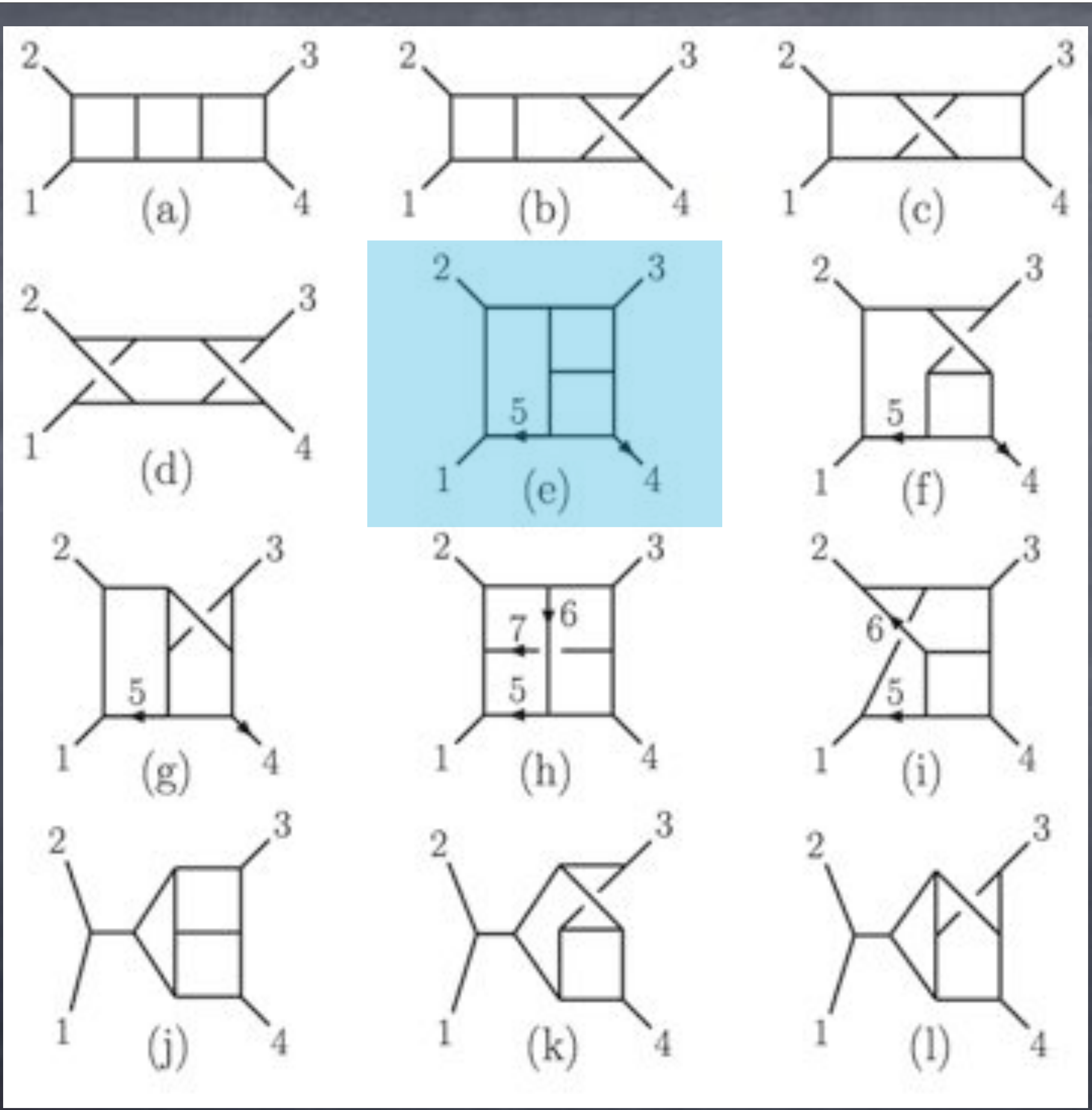
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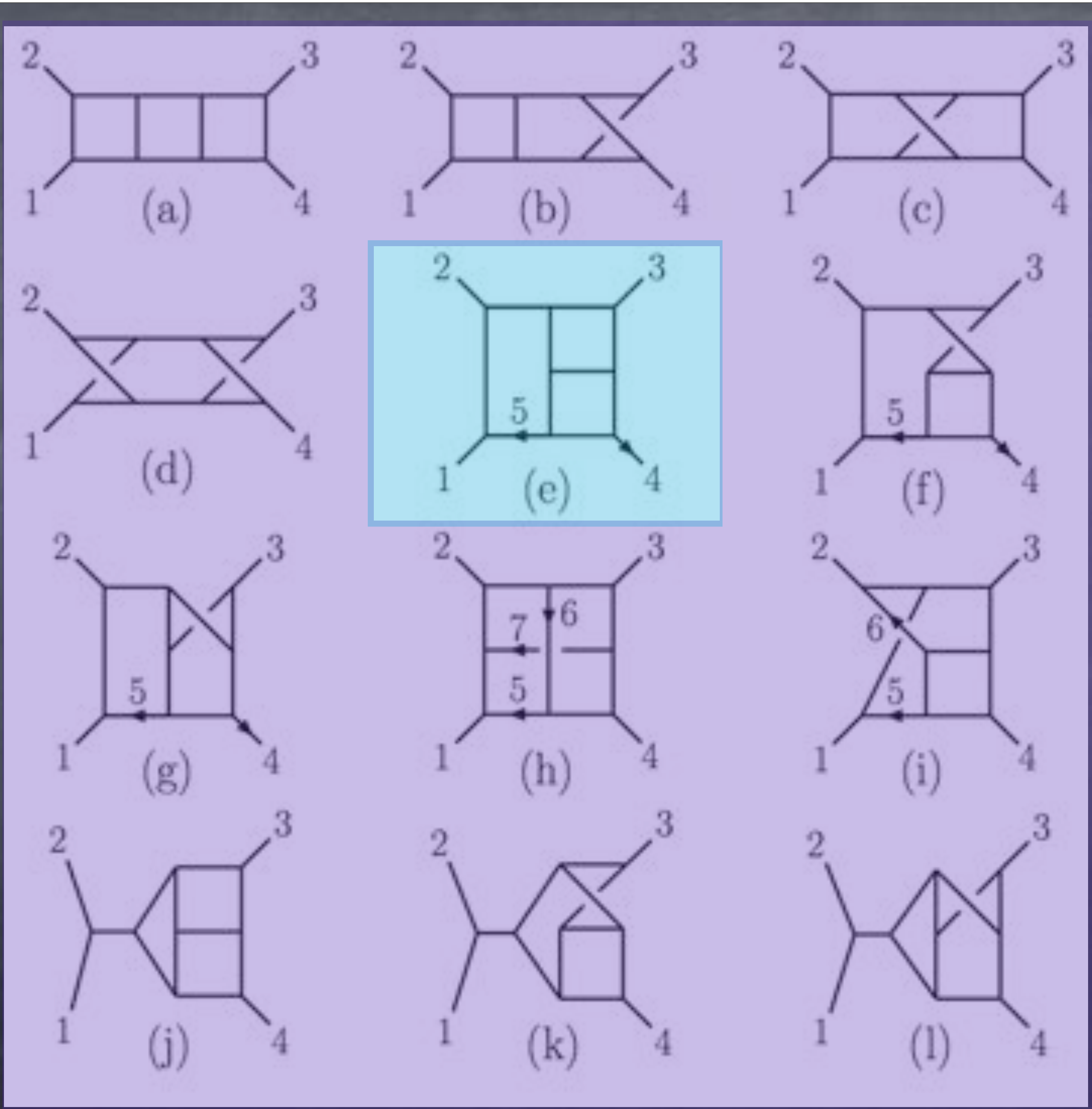
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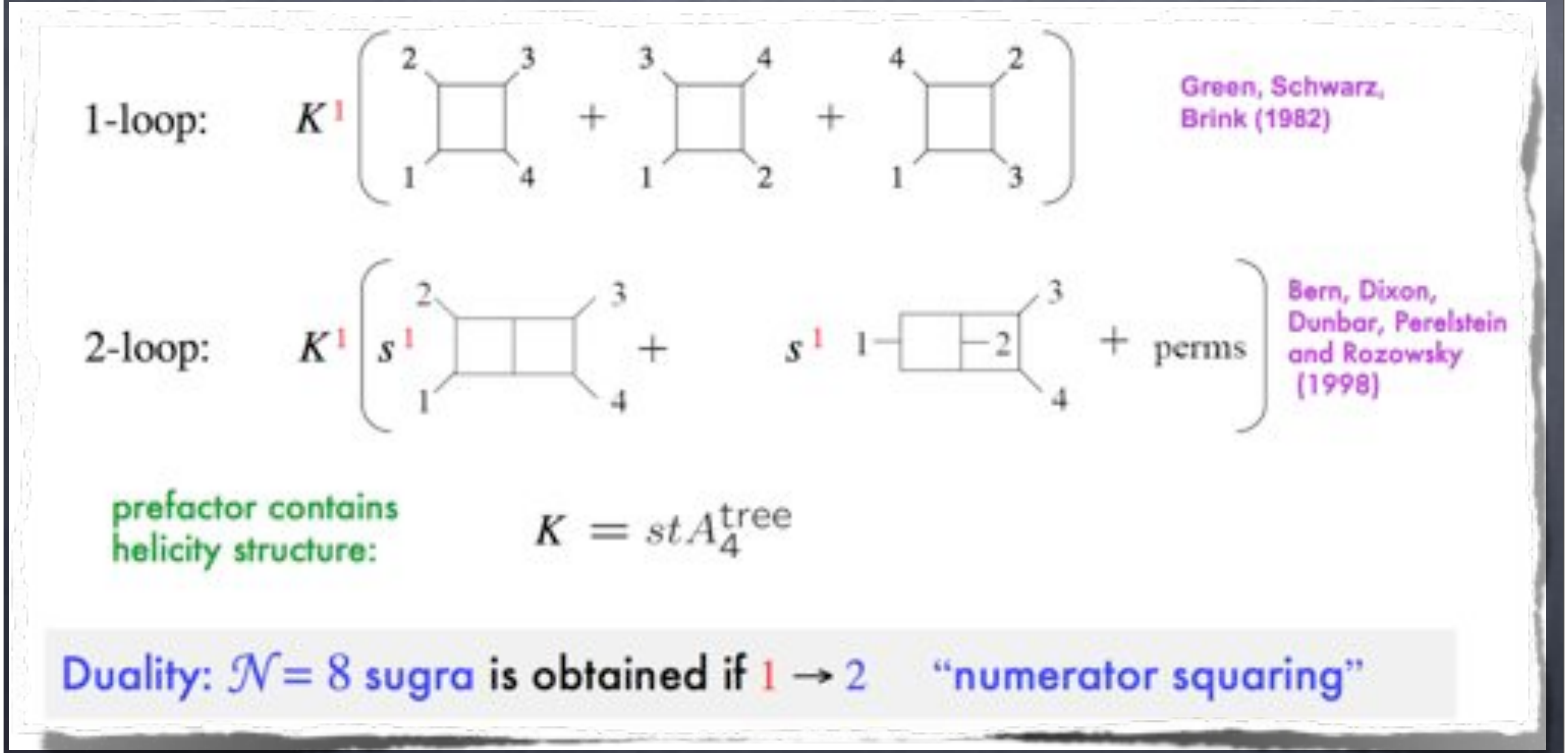
We know this works beautifully at 1 and 2 loops for N=4 and N=8!



1-loop: K^{1} (+ +)

Green, Schwarz, Brink (1982)

2-loop: K^{1} (s^{1} + s^{1} + perms)

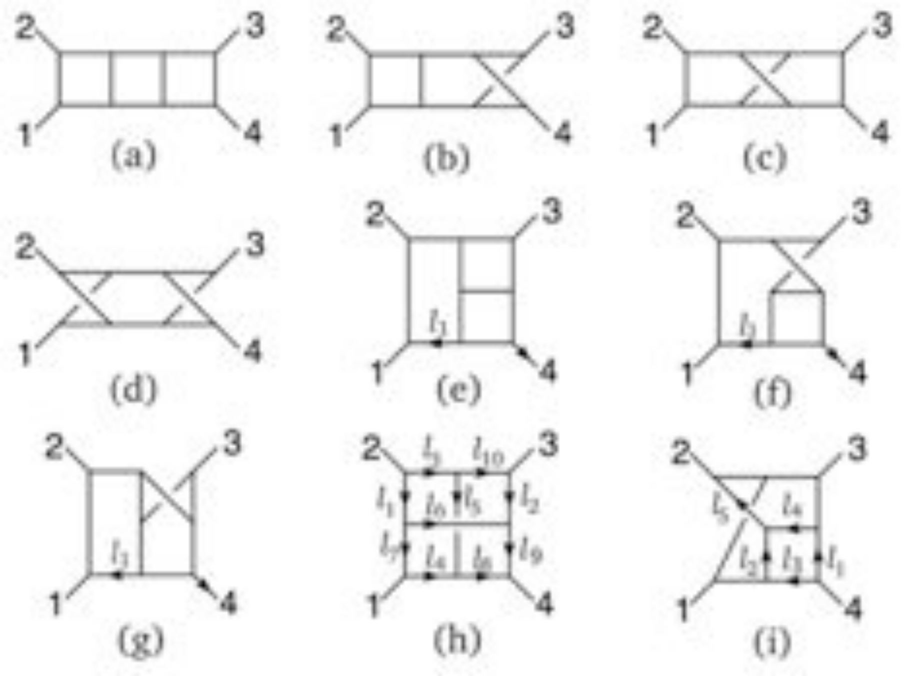
Bern, Dixon, Dunbar, Perelstein and Rozowsky (1998)

prefactor contains helicity structure:

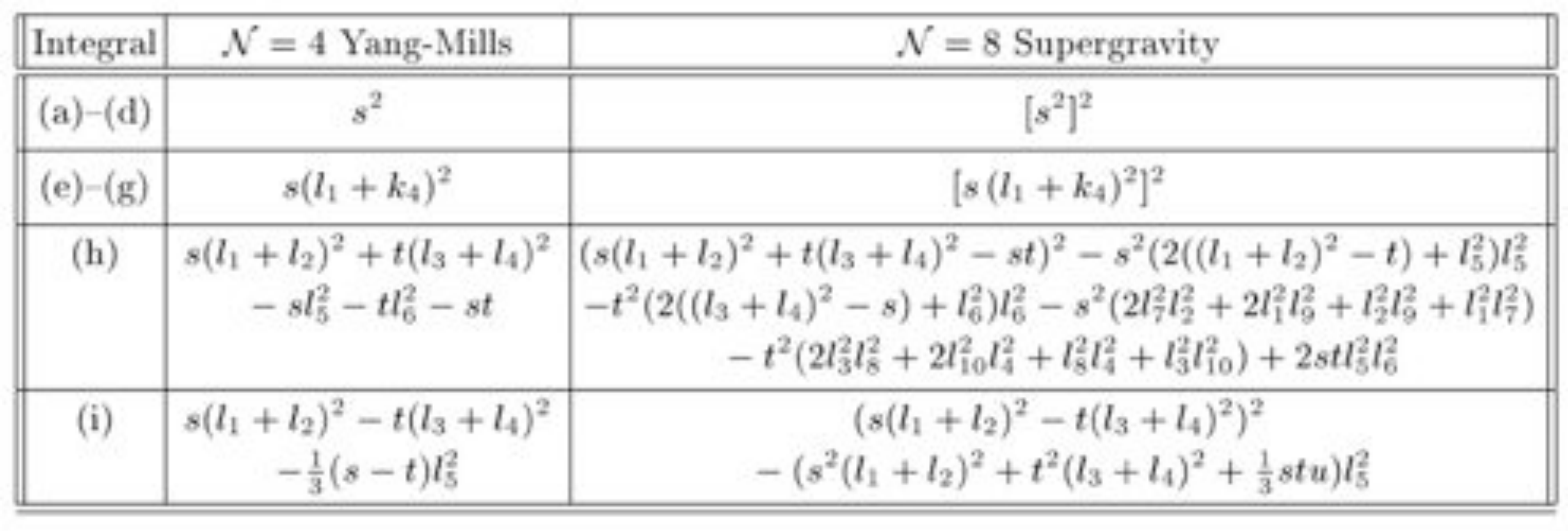
$$K = stA_4^{\text{tree}}$$

Duality: $\mathcal{N} = 8$ sugra is obtained if $1 \to 2$ "numerator squaring"

BCDJKR



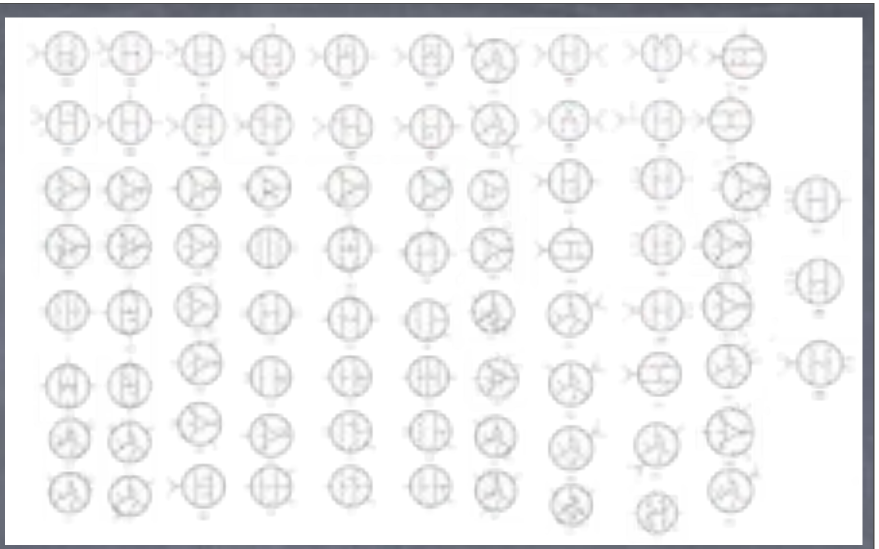
Original solution of three-loop four-point N=4 sYM and N=8 sugra

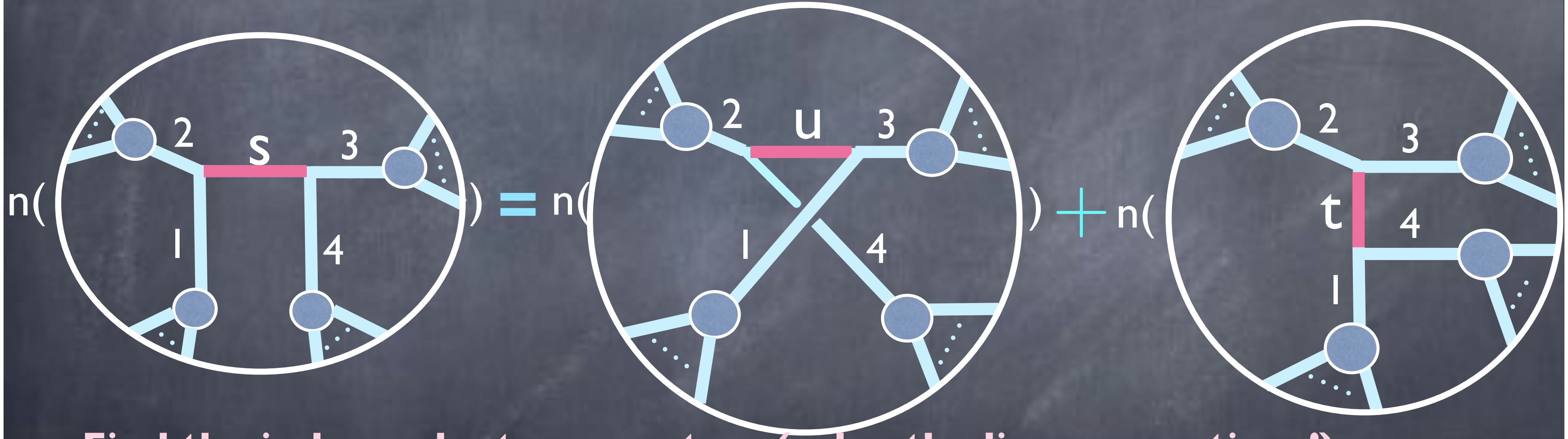


Integral	$\mathcal{N}=4$ Yang-Mills	$\mathcal{N}=8$ Supergravity
(a)–(d)	s^2	$[s^2]^2$
(e)–(g)	$s(l_1+k_4)^2$	$[s\,(l_1+k_4)^2]^2$
(h)	$s(l_1+l_2)^2+t(l_3+l_4)^2$ $-\,sl_5^2-tl_6^2-st$	$(s(l_1+l_2)^2+t(l_3+l_4)^2-st)^2-s^2(2((l_1+l_2)^2-t)+l_5^2)l_5^2$ $-t^2(2((l_3+l_4)^2-s)+l_6^2)l_6^2-s^2(2l_7^2l_2^2+2l_1^2l_9^2+l_2^2l_9^2+l_1^2l_7^2)$ $-\,t^2(2l_3^2l_8^2+2l_{10}^2l_4^2+l_8^2l_4^2+l_3^2l_{10}^2)+2stl_5^2l_6^2$
(i)	$s(l_1+l_2)^2-t(l_3+l_4)^2$ $-\frac{1}{3}(s-t)l_5^2$	$(s(l_1+l_2)^2-t(l_3+l_4)^2)^2$ $-\,(s^2(l_1+l_2)^2+t^2(l_3+l_4)^2+\frac{1}{3}stu)l_5^2$

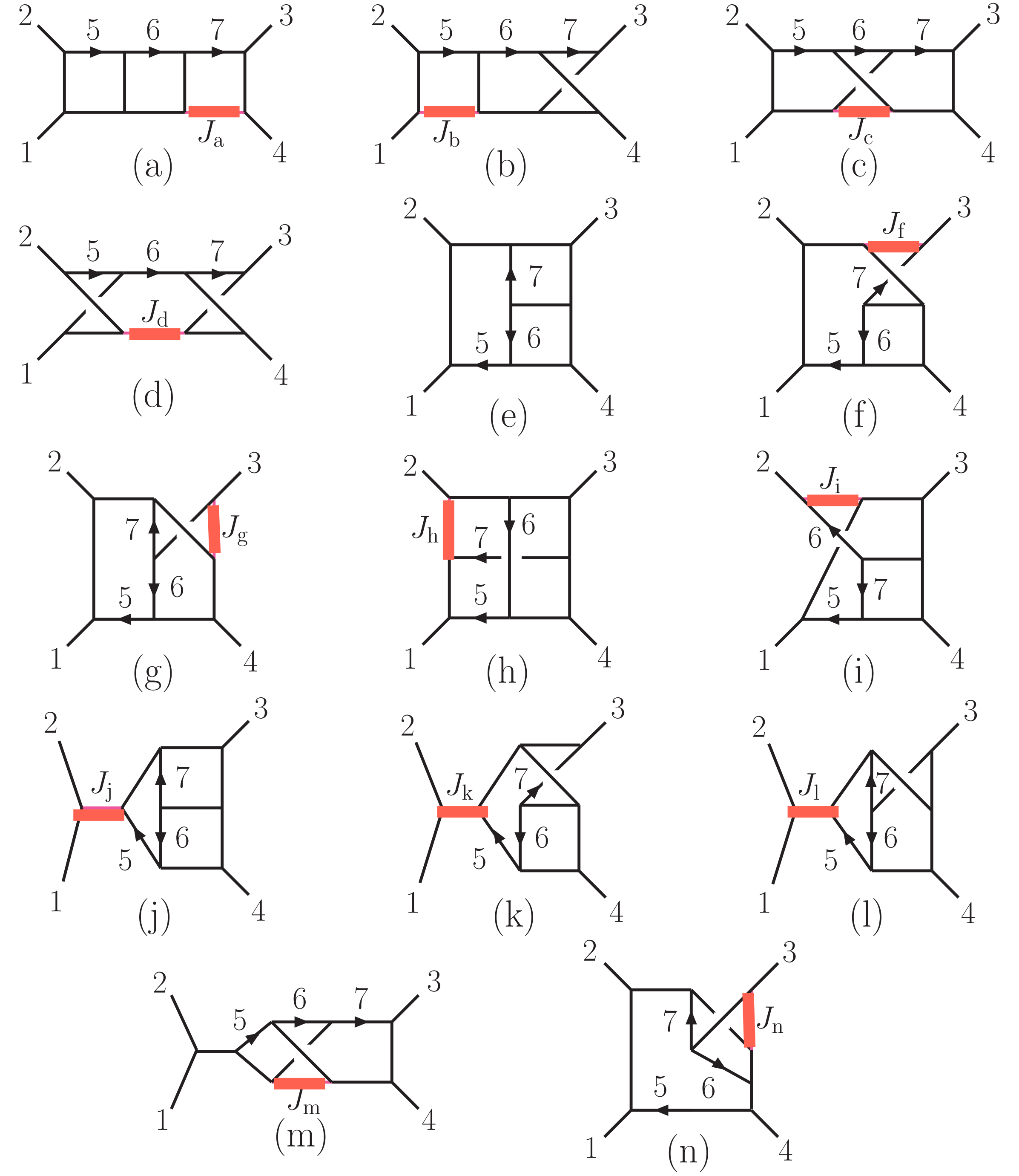
Recipe for finding $\triangle$ so dressings satisfy duality:

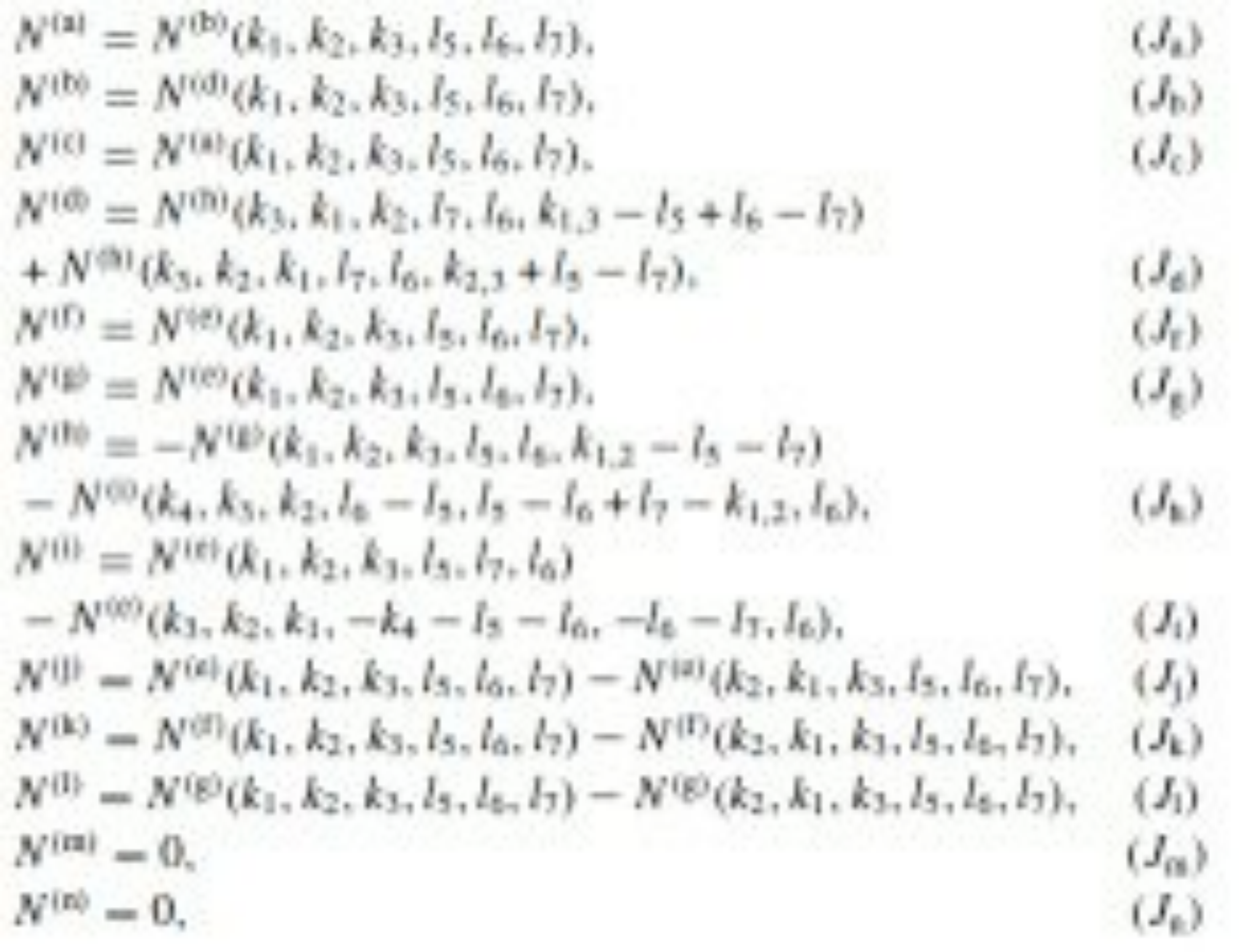
- Every edge represents a set of constraints on functional form of the numerators of the graphs. Small fraction needed.





- Find the independent numerators (solve the linear equations!)
- Build ansatze for such ``masters" graph numerators using functions seen on exploratory cuts
- Impose relevant symmetries
- Fit to the theory!


2 5 6 7 3 1 4 J_a (a)
2 5 6 7 3 1 4 J_b (b)
2 5 6 7 3 1 4 J_c (c)
2 5 6 7 3 1 4 J_d (d)
2 3 7 5 6 1 4 (e)
2 J_f 3 7 5 6 1 4 (f)
2 3 7 J_g 5 6 1 4 (g)
2 3 J_h 7 6 5 1 4 (h)
2 J_i 3 6 5 7 1 4 (i)
2 3 J_j 7 5 6 1 4 (j)
2 3 J_k 7 5 6 1 4 (k)
2 3 J_l 7 5 6 1 4 (l)
2 5 6 7 3 1 J_m 4 (m)
2 3 7 J_n 5 6 1 4 (n)



$$N^{(a)} = N^{(b)}(k_1, k_2, k_3, l_5, l_6, l_7), \qquad (J_a)$$

$$N^{(b)} = N^{(d)}(k_1, k_2, k_3, l_5, l_6, l_7), \qquad (J_b)$$

$$N^{(c)} = N^{(a)}(k_1, k_2, k_3, l_5, l_6, l_7), \qquad (J_c)$$

$$N^{(d)} = N^{(h)}(k_3, k_1, k_2, l_7, l_6, k_{1,3} - l_5 + l_6 - l_7)$$

$$+ N^{(h)}(k_3, k_2, k_1, l_7, l_6, k_{2,3} + l_5 - l_7), \qquad (J_d)$$

$$N^{(f)} = N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7), \qquad (J_f)$$

$$N^{(g)} = N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7), \qquad (J_g)$$

$$N^{(h)} = -N^{(g)}(k_1, k_2, k_3, l_5, l_6, k_{1,2} - l_5 - l_7)$$

$$- N^{(i)}(k_4, k_3, k_2, l_6 - l_5, l_5 - l_6 + l_7 - k_{1,2}, l_6), \qquad (J_h)$$

$$N^{(i)} = N^{(e)}(k_1, k_2, k_3, l_5, l_7, l_6)$$

$$- N^{(e)}(k_3, k_2, k_1, -k_4 - l_5 - l_6, -l_6 - l_7, l_6), \qquad (J_i)$$

$$N^{(j)} = N^{(e)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(e)}(k_2, k_1, k_3, l_5, l_6, l_7), \qquad (J_j)$$

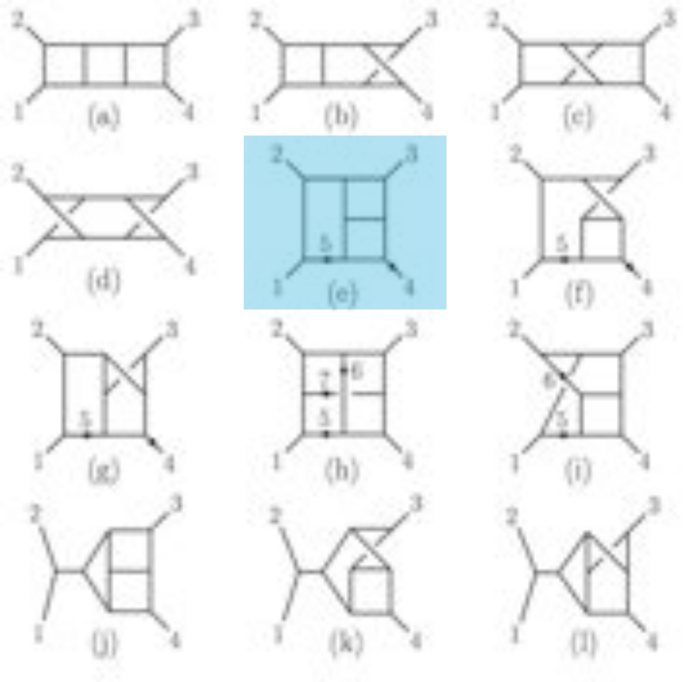
$$N^{(k)} = N^{(f)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(f)}(k_2, k_1, k_3, l_5, l_6, l_7), \qquad (J_k)$$

$$N^{(l)} = N^{(g)}(k_1, k_2, k_3, l_5, l_6, l_7) - N^{(g)}(k_2, k_1, k_3, l_5, l_6, l_7), \qquad (J_l)$$

$$N^{(m)} = 0, \qquad (J_m)$$

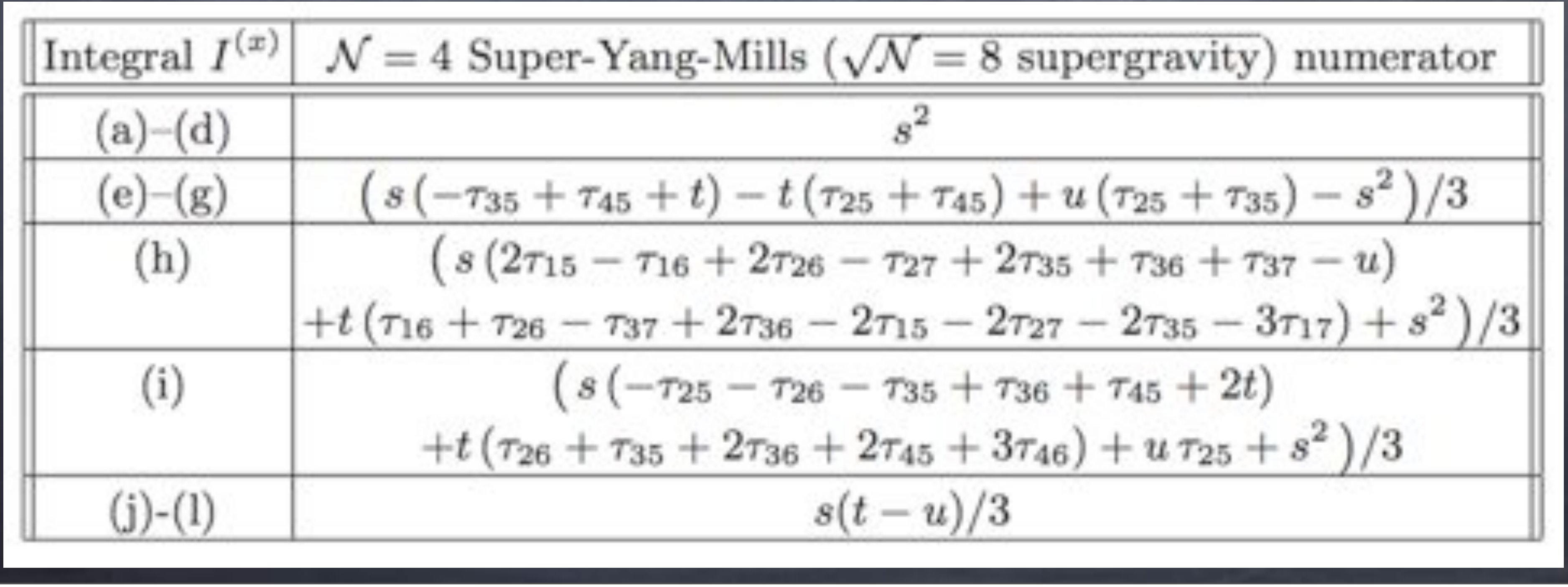
$$N^{(n)} = 0, \qquad (J_n)$$

BCJ (2010)



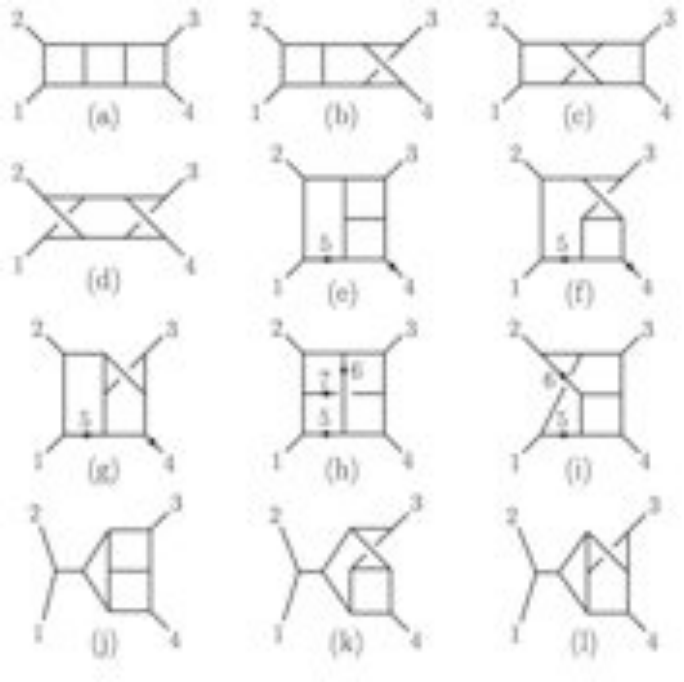
Only, e.g., require maximal cut information of (e) graph to build full amplitude!

$s=(k_1+k_2)^2 \quad t=(k_1+k_4)^2 \quad u=(k_1+k_3)^2 \quad \tau_{i,j}=2k_i\cdot l_j$



Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8}$ supergravity) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\left(-\tau_{35}+\tau_{45}+t\right)-t\left(\tau_{25}+\tau_{45}\right)+u\left(\tau_{25}+\tau_{35}\right)-s^2\right)/3$
(h)	$\left(s\left(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u\right)\right.$ $\left.+t\left(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}\right)+s^2\right)/3$
(i)	$\left(s\left(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t\right)\right.$ $\left.+t\left(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}\right)+u\,\tau_{25}+s^2\right)/3$
(j)-(l)	$s(t-u)/3$

BCJ (2010)

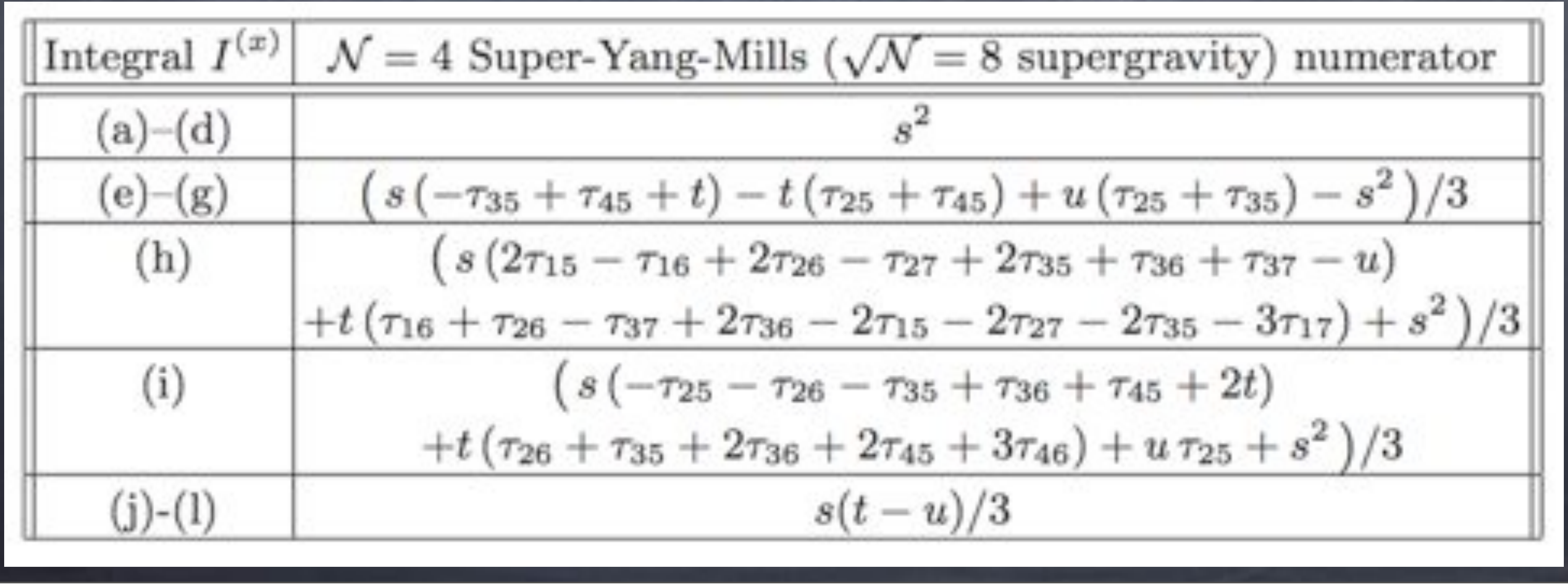


Note:

BOTH N=4 sYM and N=8 sugra

manifestly have same overall powercounting!

$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2 \quad u = (k_1 + k_3)^2 \quad \tau_{i,j} = 2k_i \cdot l_j$

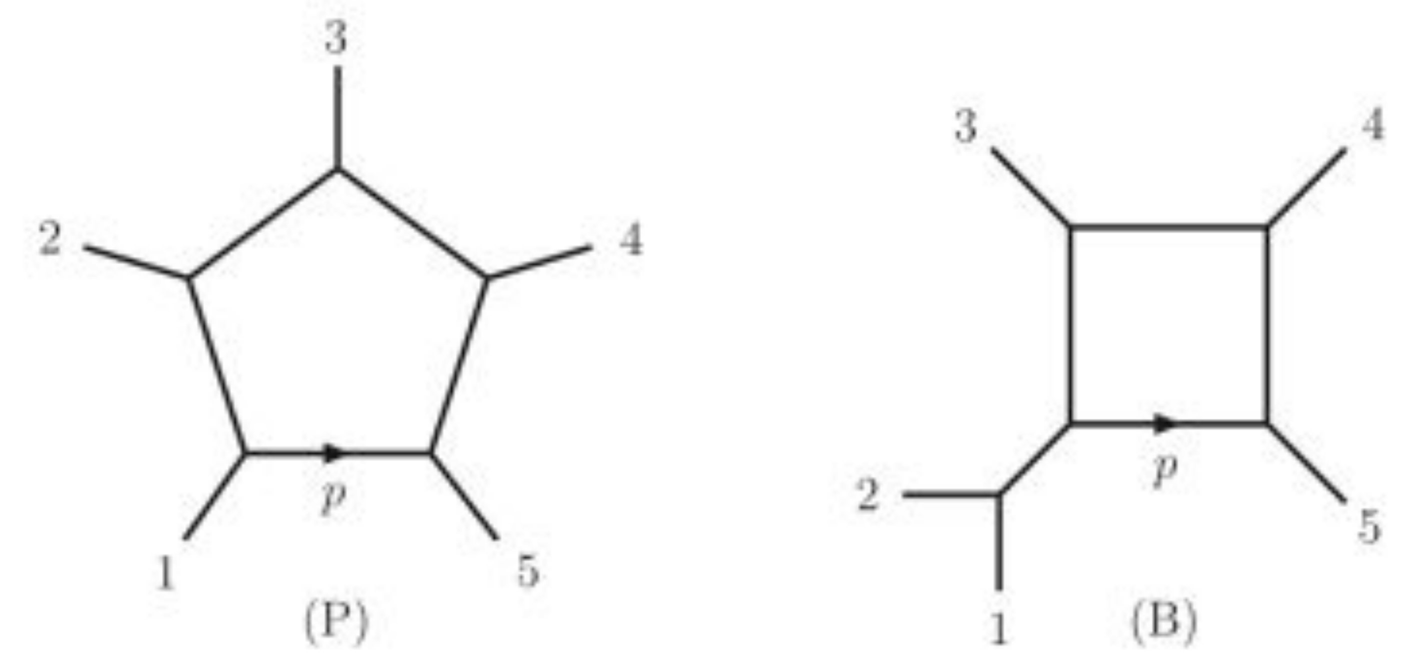


Integral $I^{(x)}$	$\mathcal{N} = 4$ Super-Yang-Mills ($\sqrt{\mathcal{N} = 8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\big(s\,(-\tau_{35} + \tau_{45} + t) - t\,(\tau_{25} + \tau_{45}) + u\,(\tau_{25} + \tau_{35}) - s^2\big)/3$
(h)	$\big(s\,(2\tau_{15} - \tau_{16} + 2\tau_{26} - \tau_{27} + 2\tau_{35} + \tau_{36} + \tau_{37} - u)$ $+t\,(\tau_{16} + \tau_{26} - \tau_{37} + 2\tau_{36} - 2\tau_{15} - 2\tau_{27} - 2\tau_{35} - 3\tau_{17}) + s^2\big)/3$
(i)	$\big(s\,(-\tau_{25} - \tau_{26} - \tau_{35} + \tau_{36} + \tau_{45} + 2t)$ $+t\,(\tau_{26} + \tau_{35} + 2\tau_{36} + 2\tau_{45} + 3\tau_{46}) + u\,\tau_{25} + s^2\big)/3$
(j)-(l)	$s(t - u)/3$

Other Loop Level Examples

JJMC, Johansson (2011)

Five point 1-loop N=4 SYM & N=8 SUGRA

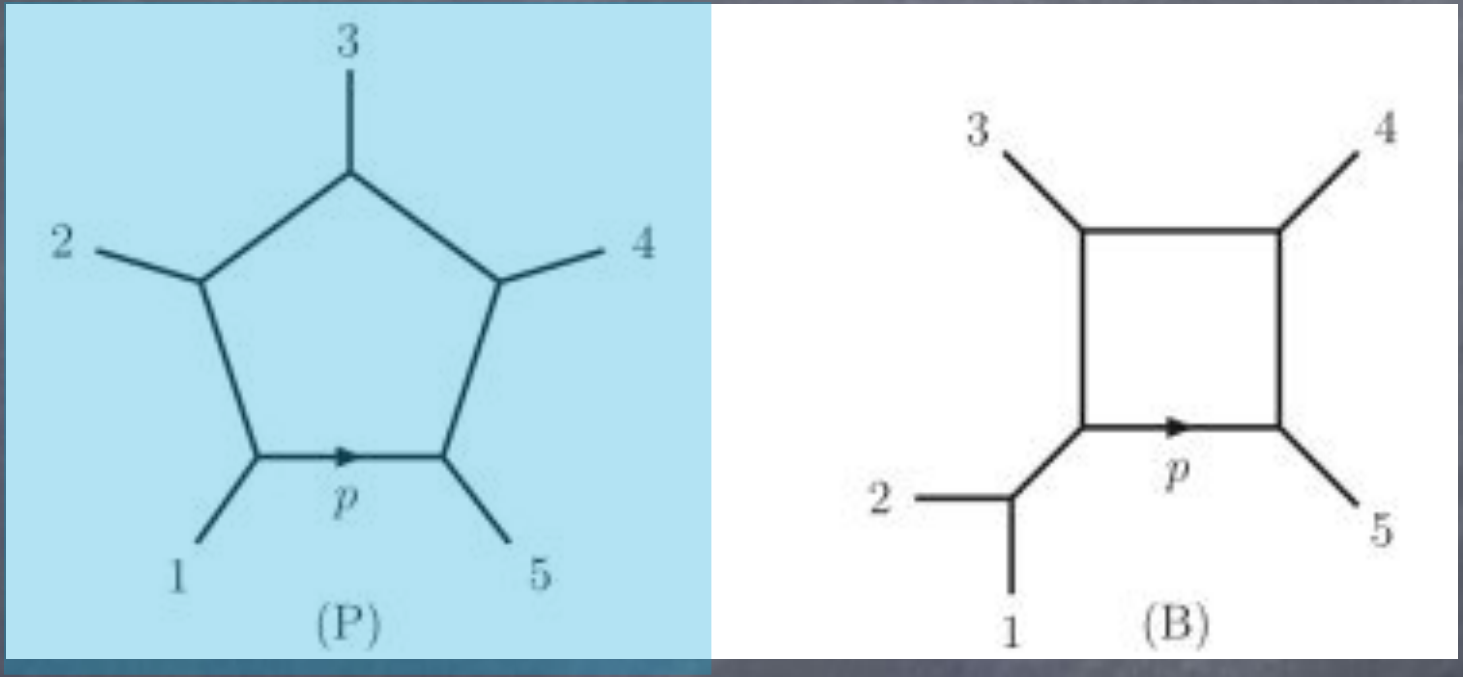


Venerable form satisfies duality (no freedom)

Bern, Dixon, Dunbar, Kosower;
Cachazo

JJMC, Johansson (2011)

Five point 1-loop N=4 SYM & N=8 SUGRA



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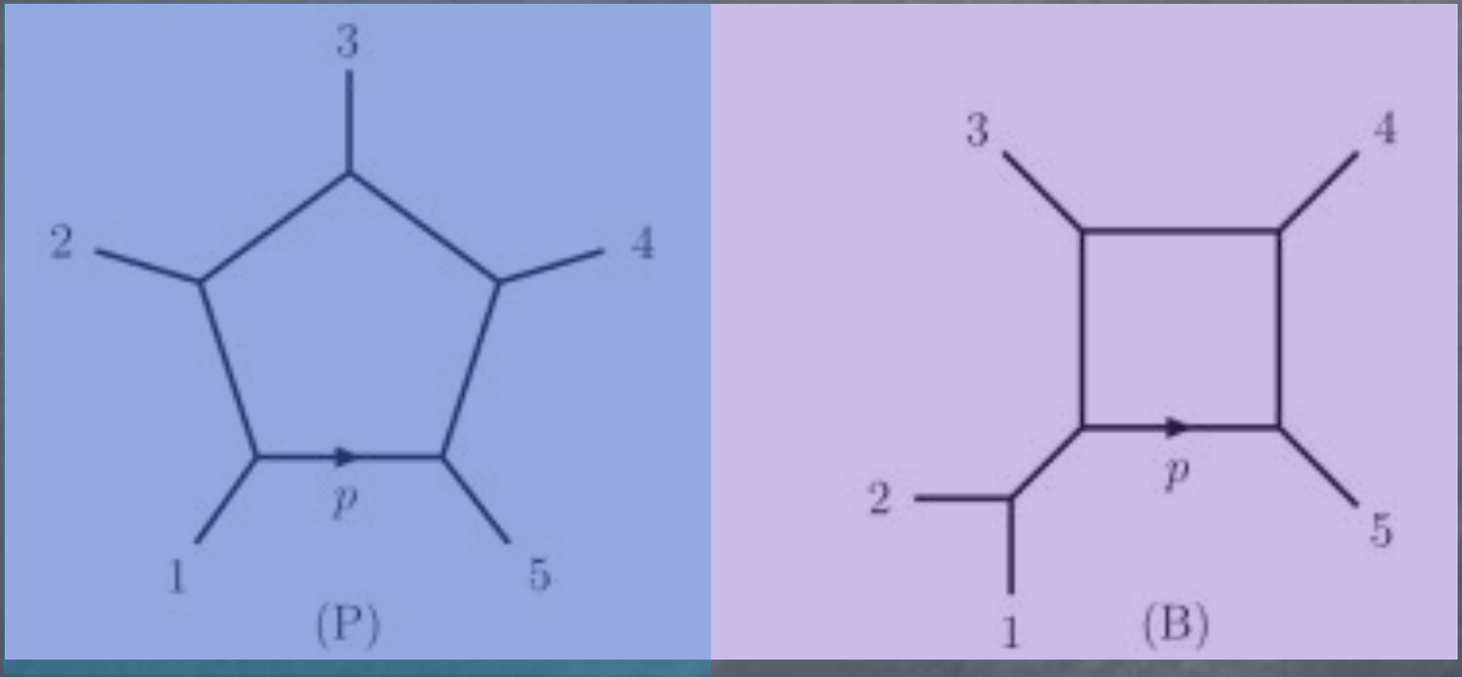
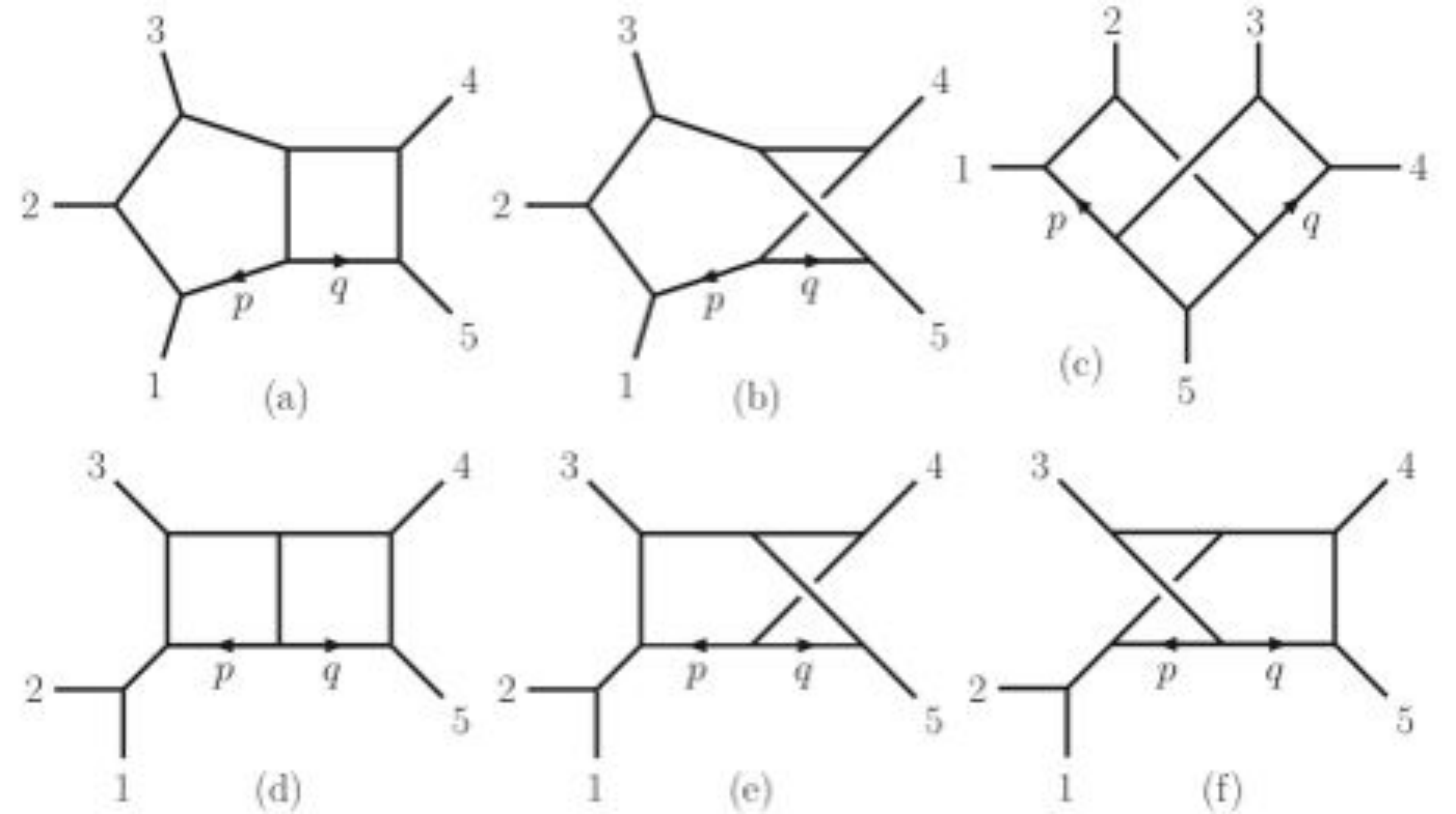


Venerable form satisfies duality (no freedom)

Bern, Dixon, Dunbar, Kosower;
Cachazo

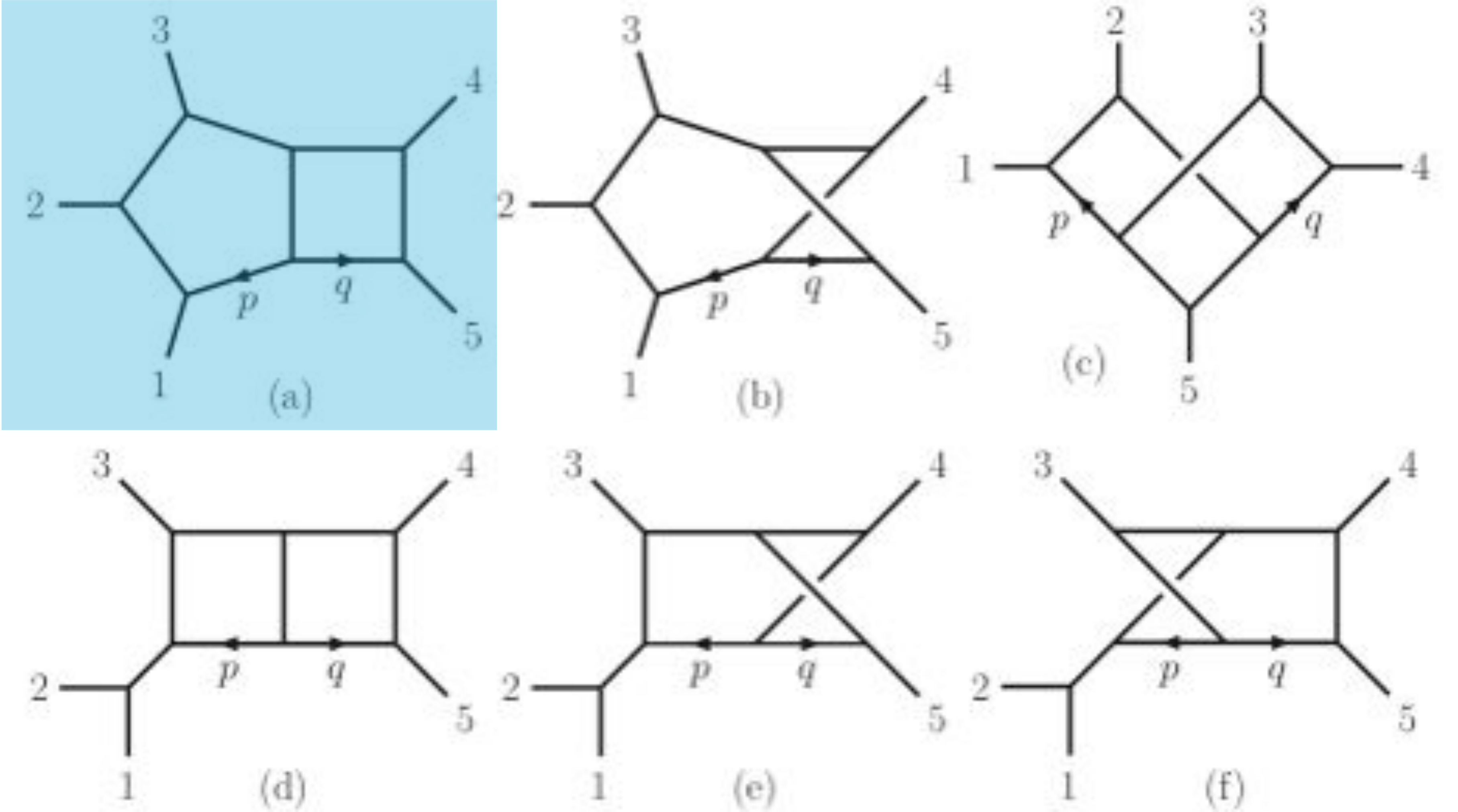
JJMC, Johansson (2011)

Five point 2-loop N=4 SYM & N=8 SUGRA



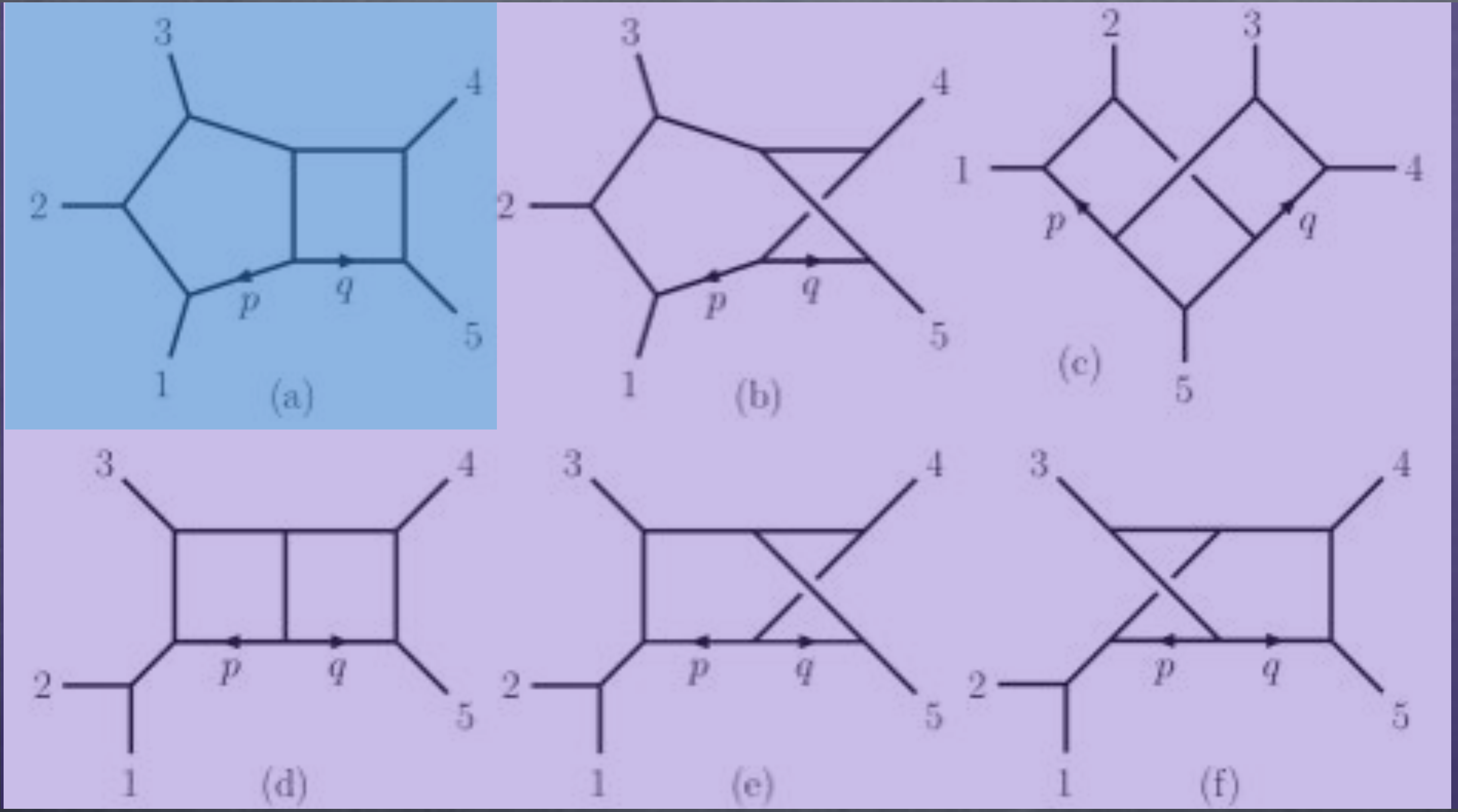
JJMC, Johansson (2011)

Five point 2-loop N=4 SYM & N=8 SUGRA



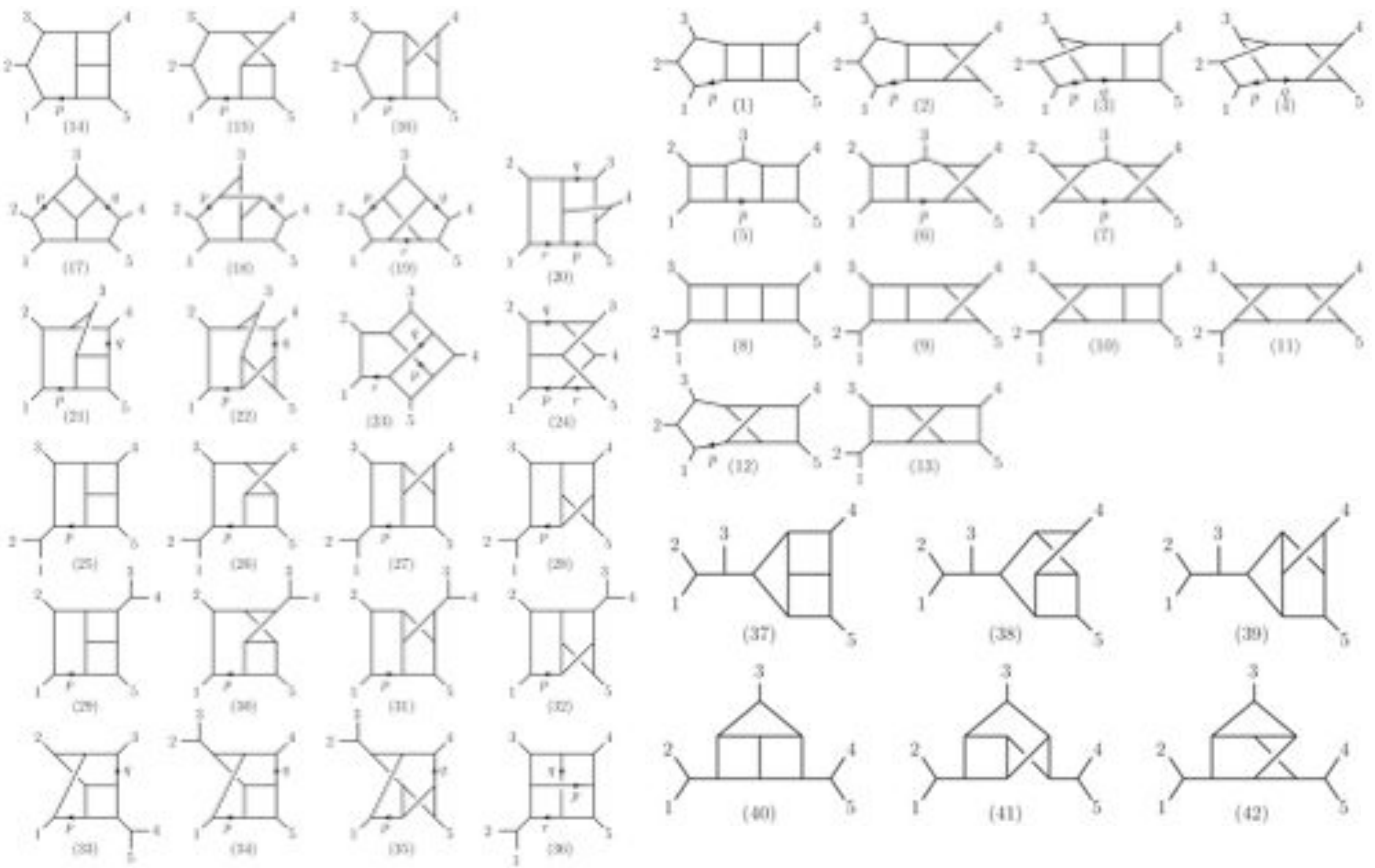
JJMC, Johansson (2011)

Five point 2-loop N=4 SYM & N=8 SUGRA



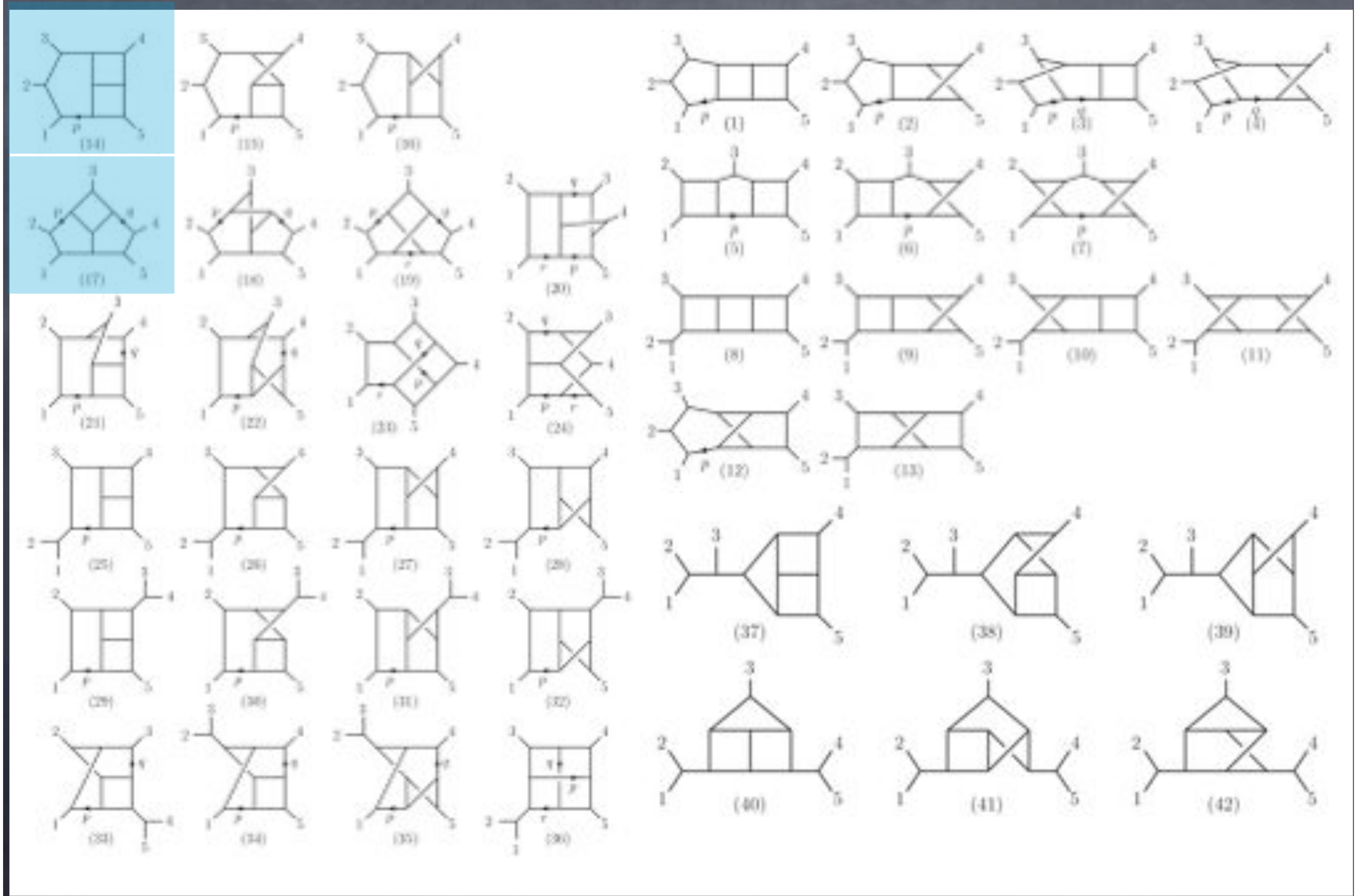
Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



Five point 3-loop N=4 SYM & N=8 SUGRA

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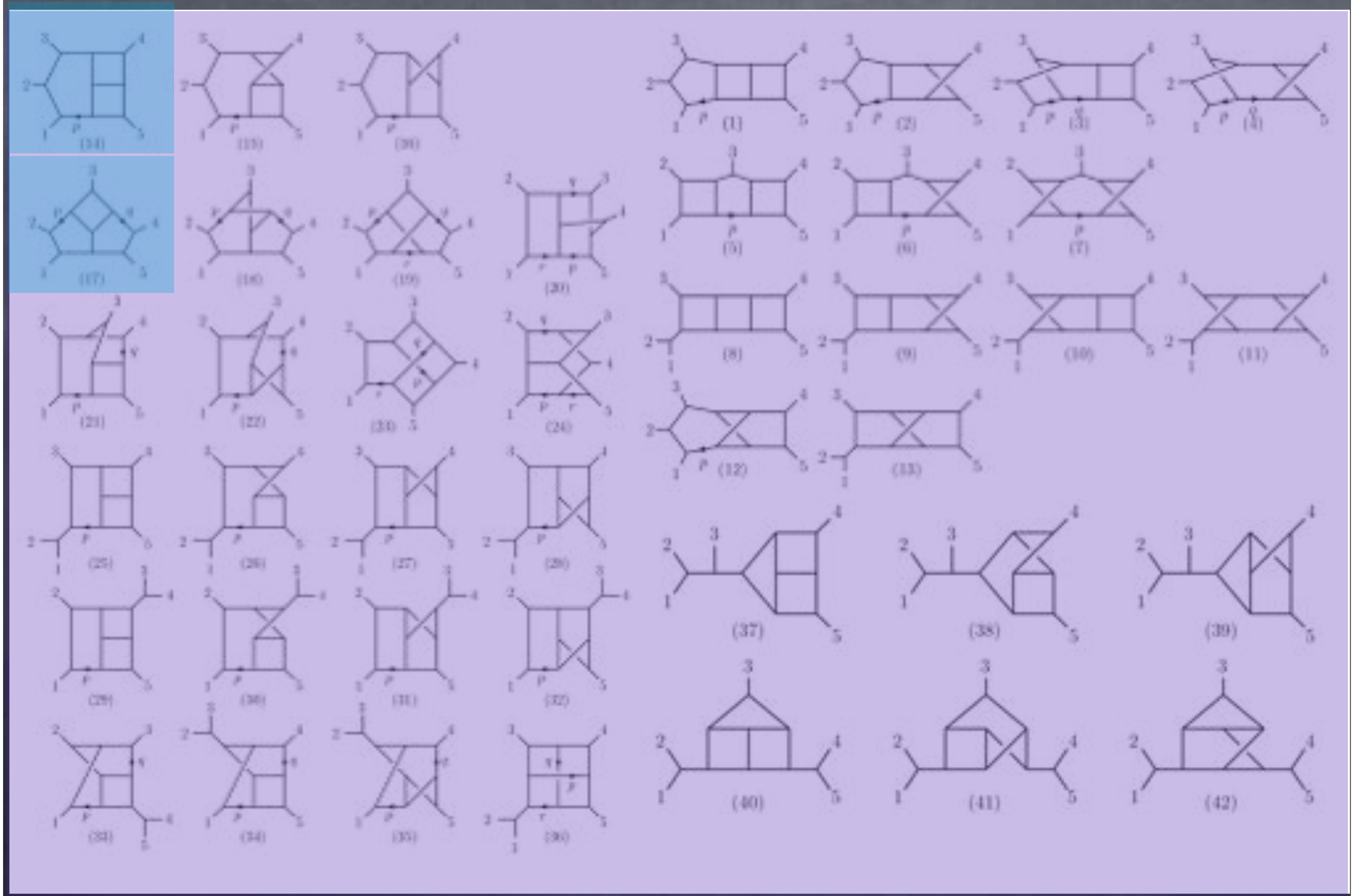
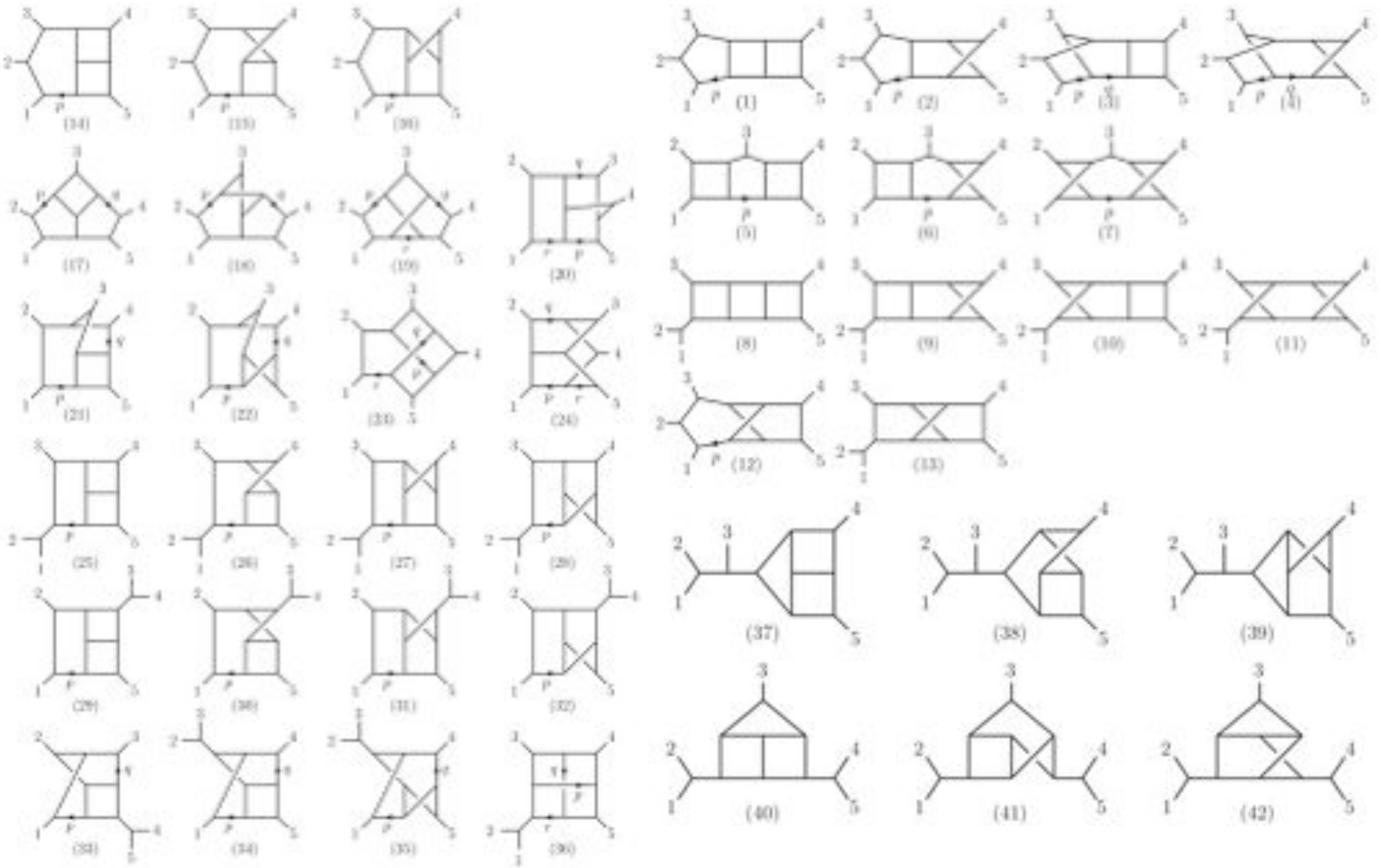


Five point 3-loop N=4 SYM & N=8 SUGRA

JJMC, Johansson (to appear)



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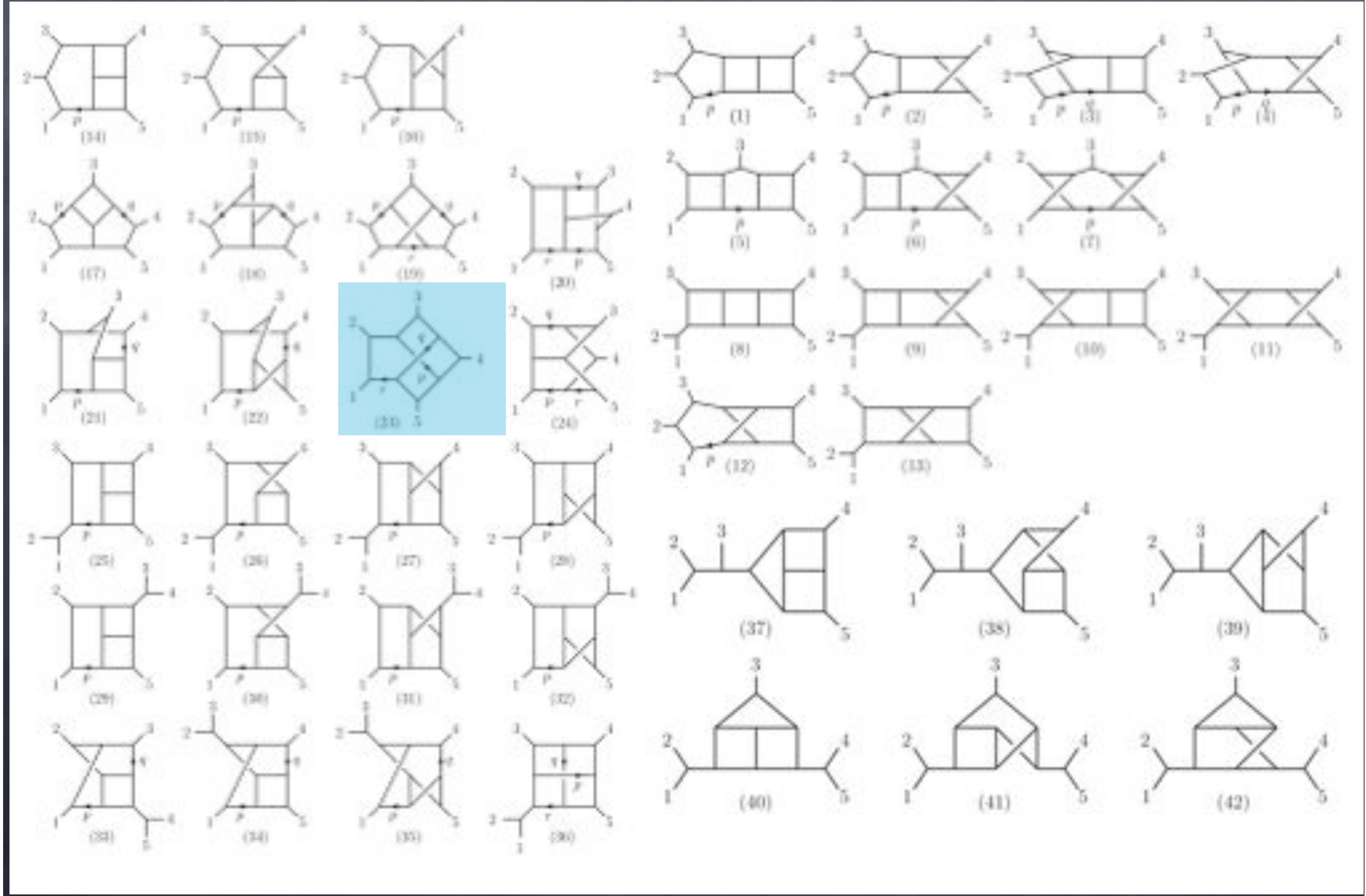

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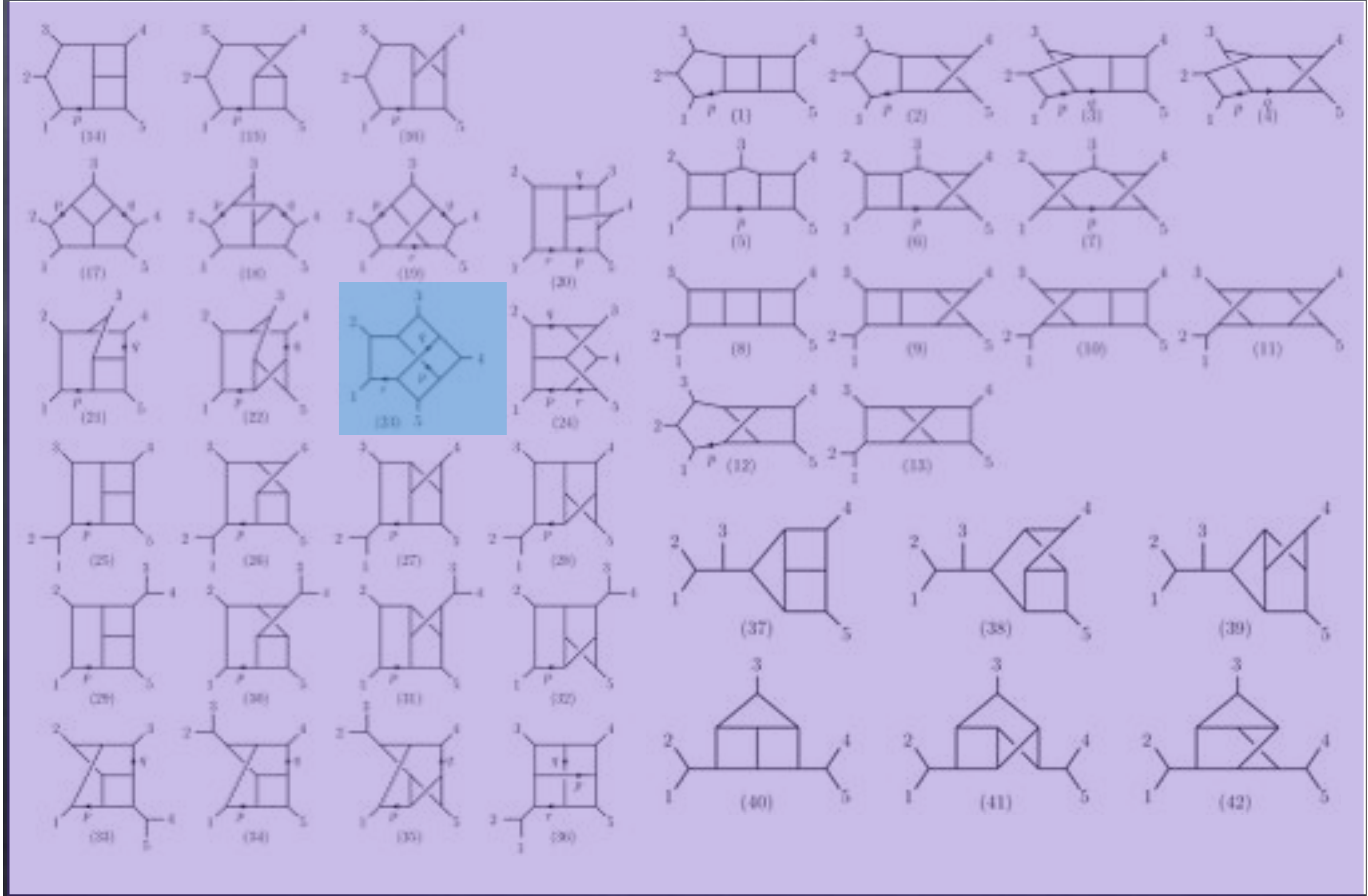
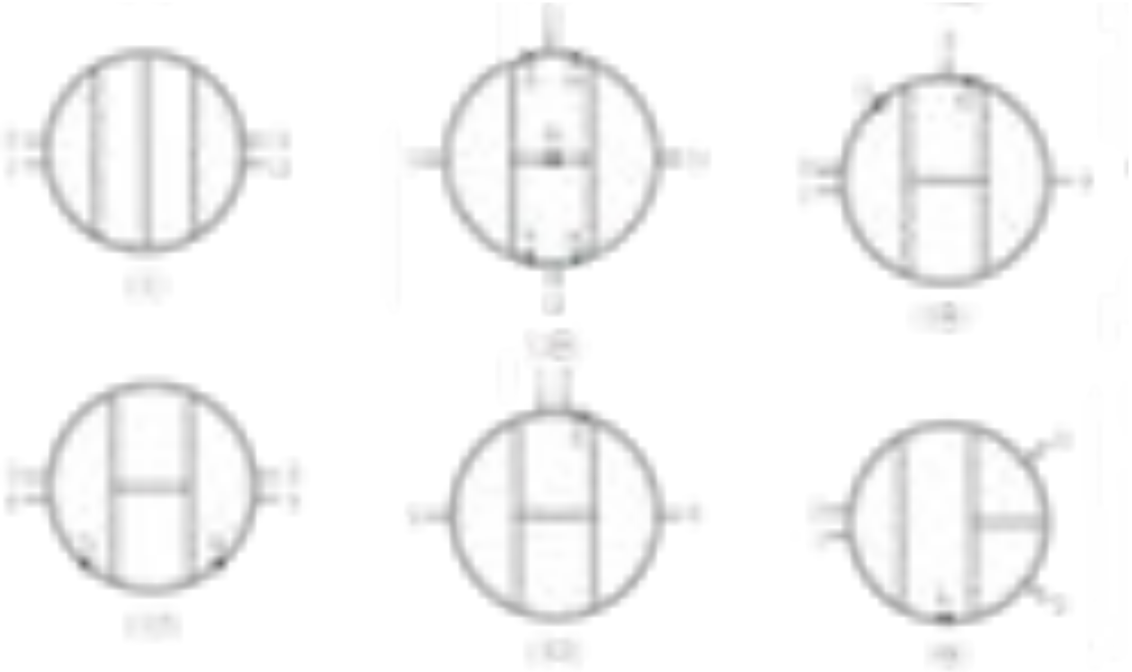


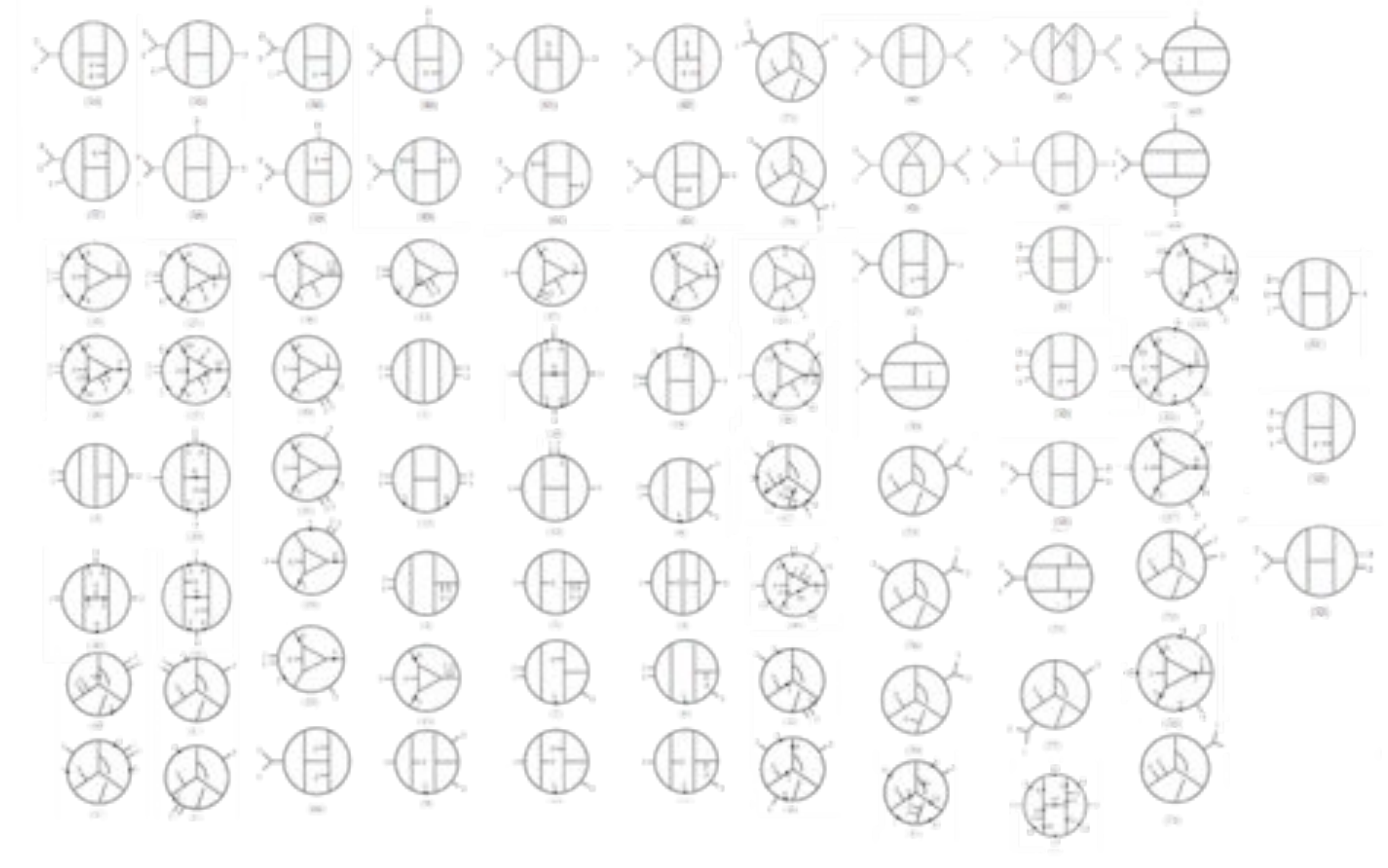
Four loop planar (extracted cusp anom. dim)



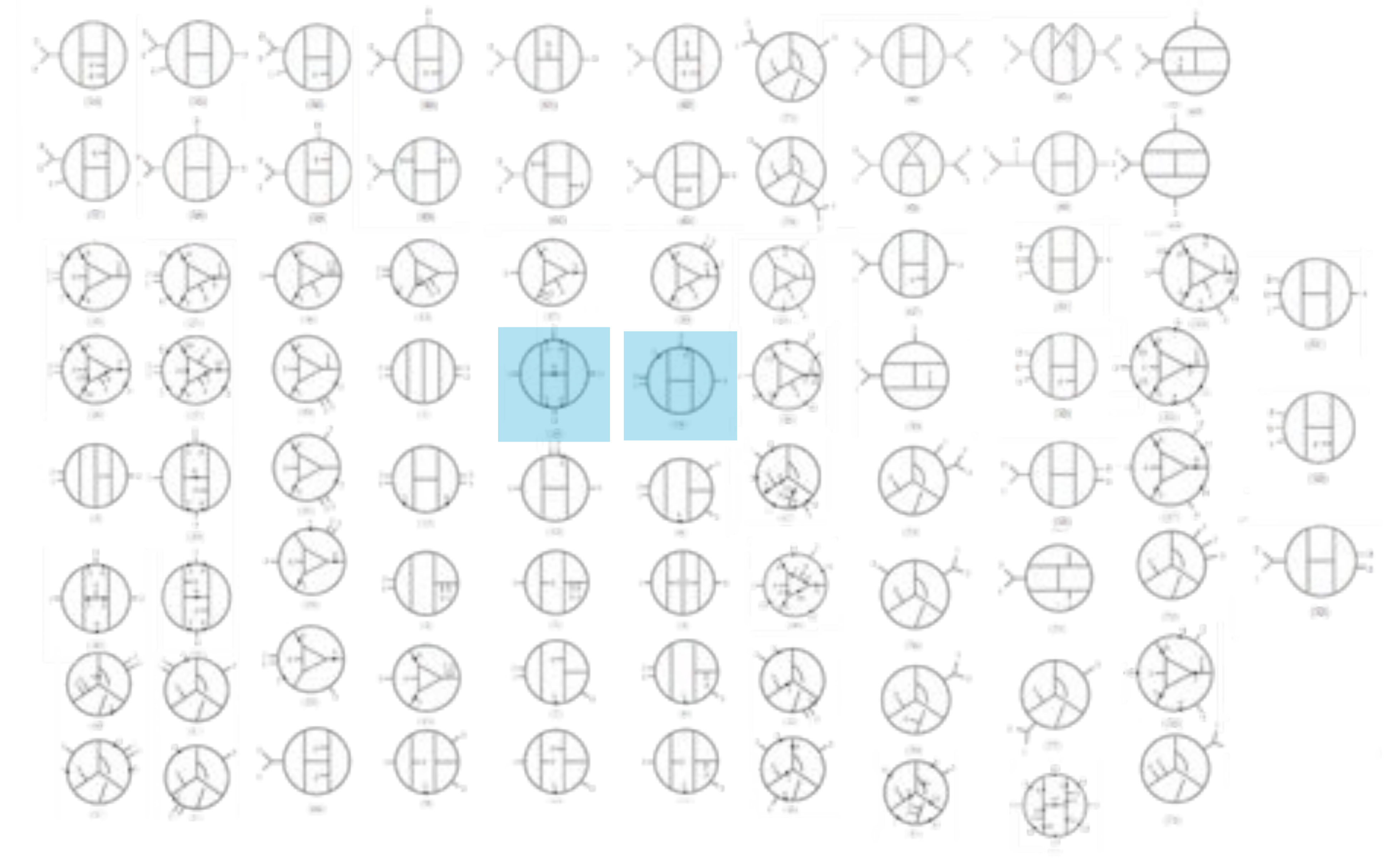
Bern, Czakon, Dixon, Kosower, Smirnov (2006)

Full four loop N=4 SYM & N=8 SUGRA

Bern, JJMC, Dixon, Johansson, Roiban (2012)

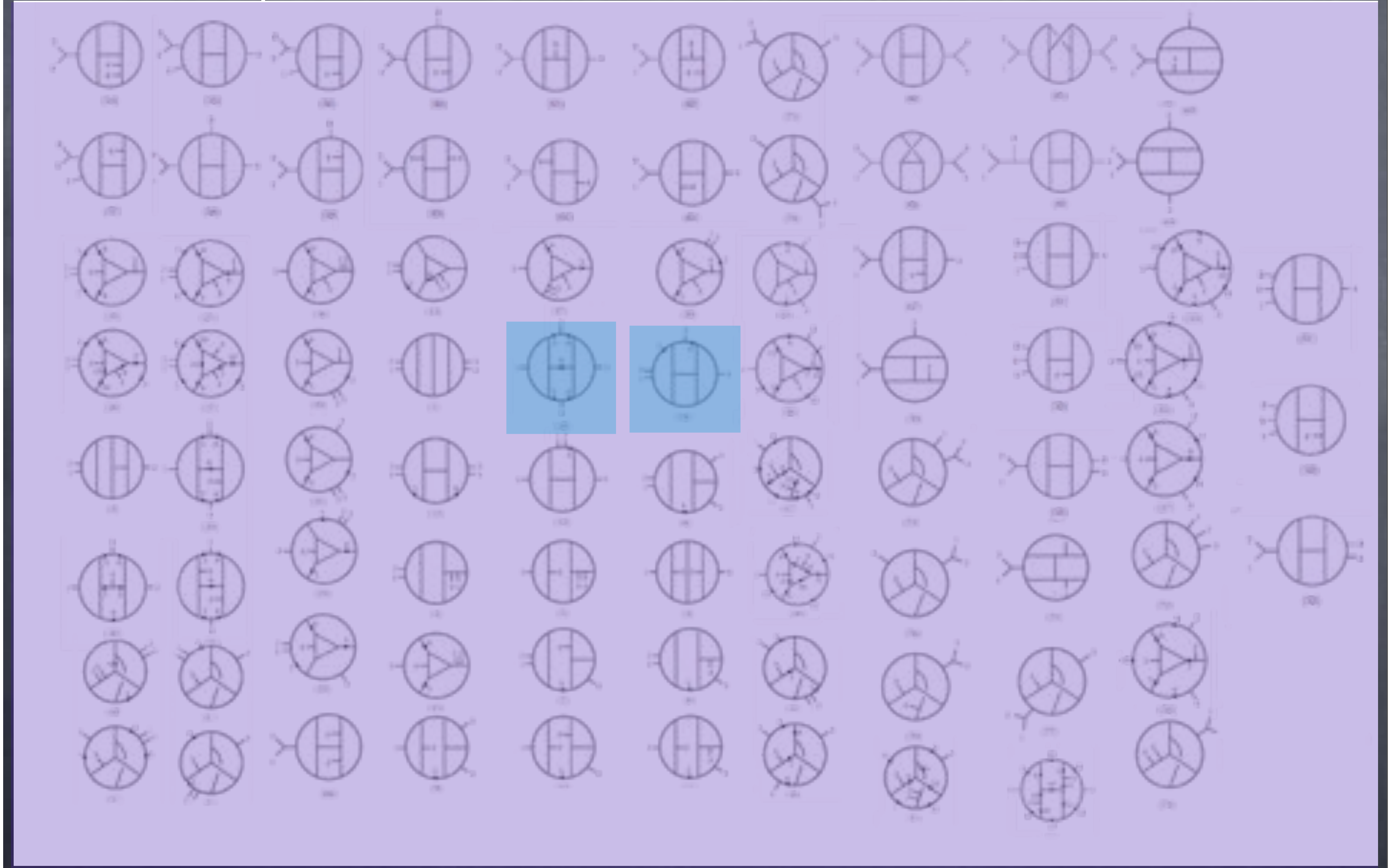


Full four loop N=4 SYM & N=8 SUGRA **Bern, JJMC, Dixon, Johansson, Roiban (2012)**

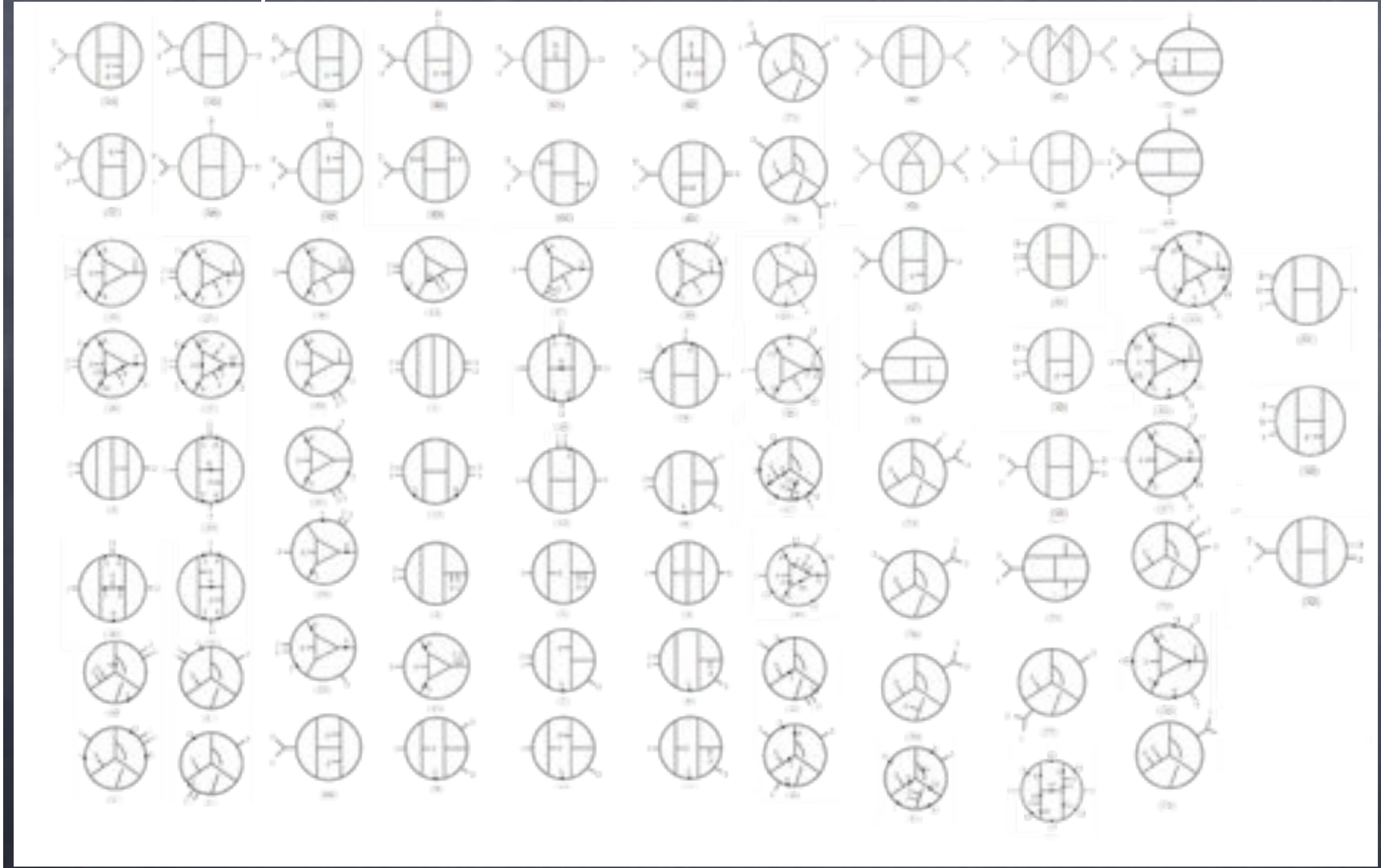


Full four loop N=4 SYM & N=8 SUGRA

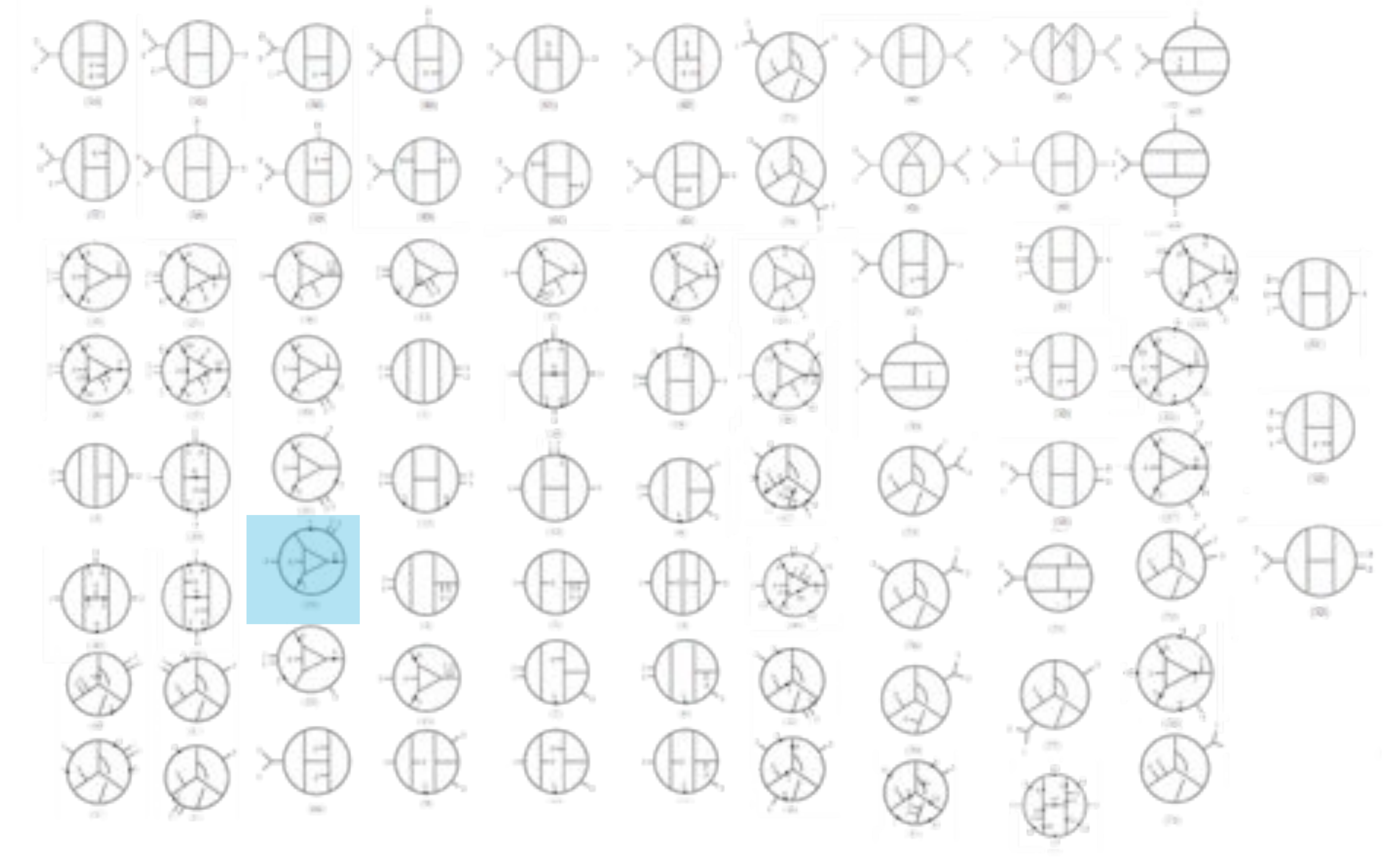
Bern, JJMC, Dixon, Johansson, Roiban (2012)



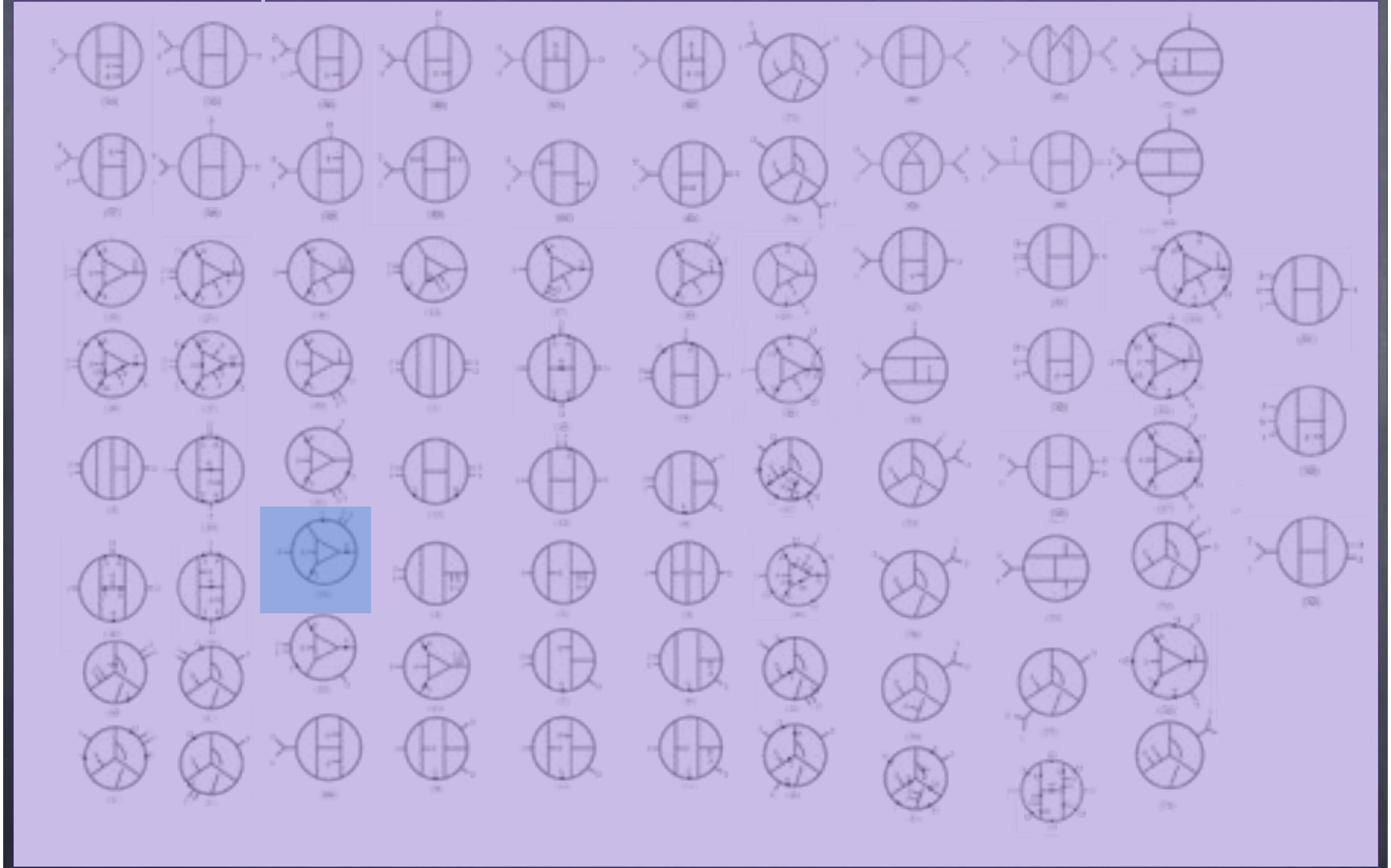
Full four loop N=4 SYM & N=8 SUGRA **Bern, JJMC, Dixon, Johansson, Roiban (2012)**



Full four loop N=4 SYM & N=8 SUGRA **Bern, JJMC, Dixon, Johansson, Roiban (2012)**



Full four loop N=4 SYM & N=8 SUGRA **Bern, JJMC, Dixon, Johansson, Roiban (2012)**



Integrated Amplitudes

$$\sum_{\mathcal{G}\in\text{cubic}} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S(\mathcal{G})} \frac{n(\mathcal{G})\tilde{n}(\mathcal{G})}{D(\mathcal{G})}$$

Note n and $\tilde{n}$ can come from different reps of same theory, or even different theories altogether.

$$\mathcal{N}=4 \text{ sYM} \otimes \mathcal{N}=4 \text{ sYM} \Rightarrow \mathcal{N}=8 \text{ sugra}$$

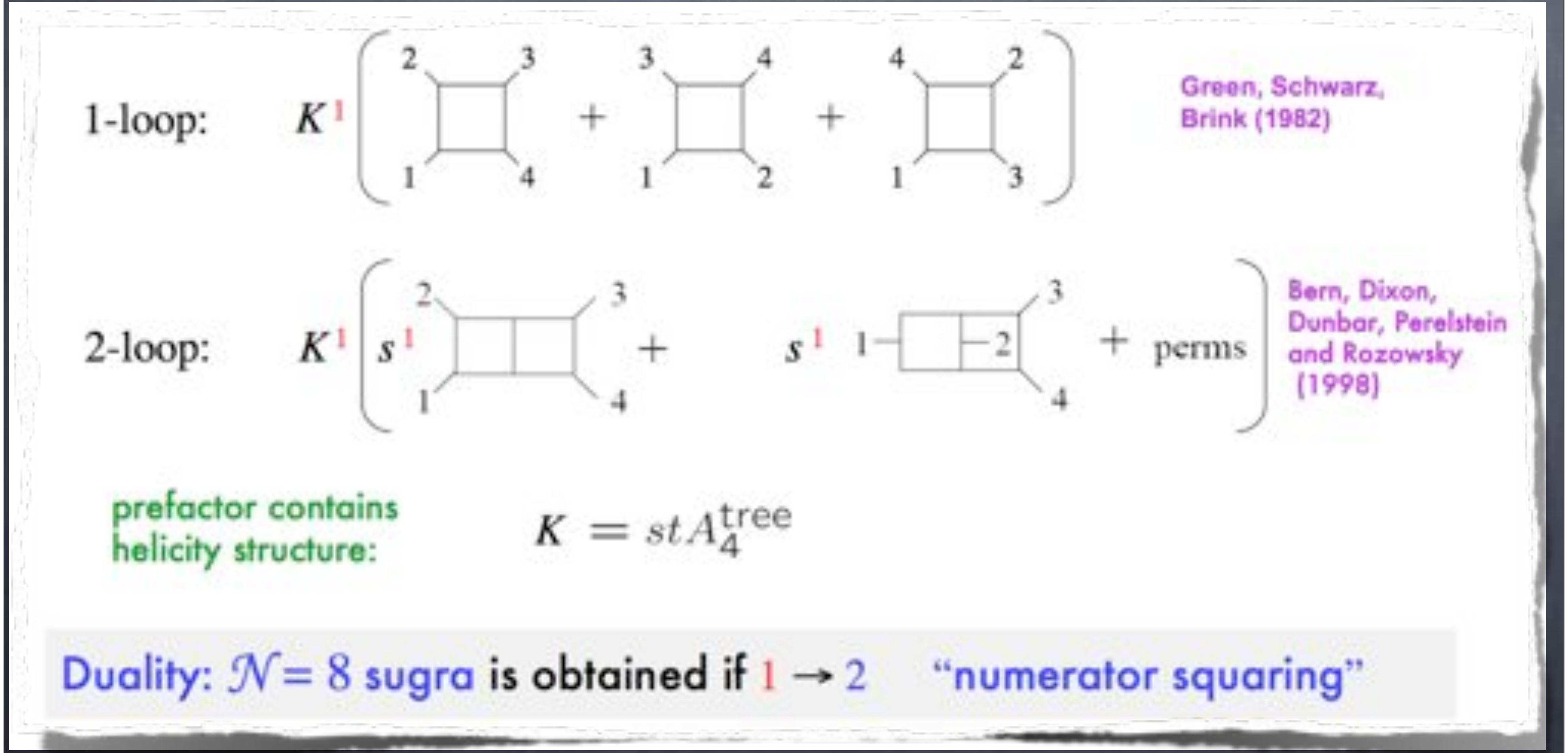
$$\mathcal{N}=p\ sYM \otimes \mathcal{N}=4 \text{ sYM} \Rightarrow \mathcal{N}=4+p \text{ sugra}$$

Only one gauge representation need have duality imposed, consequence of general freedom:

$$n(\mathcal{G}) \to n(\mathcal{G}) + \Delta(\mathcal{G}), \quad \sum_{\mathcal{G}\in\text{cubic}} \left(\frac{c(\mathcal{G})\Delta(\mathcal{G})}{D(\mathcal{G})}\right) = 0$$

can only depend on algebraic property of $c(\mathcal{G})$ not numeric values. So as long as $\tilde{n}(\mathcal{G})$ satisfies same algebra (i.e. duality) can shift $n(\mathcal{G})$ as we please.

Recall 1 & 2 Loop 4-point



1-loop: K^{1} (+ +)

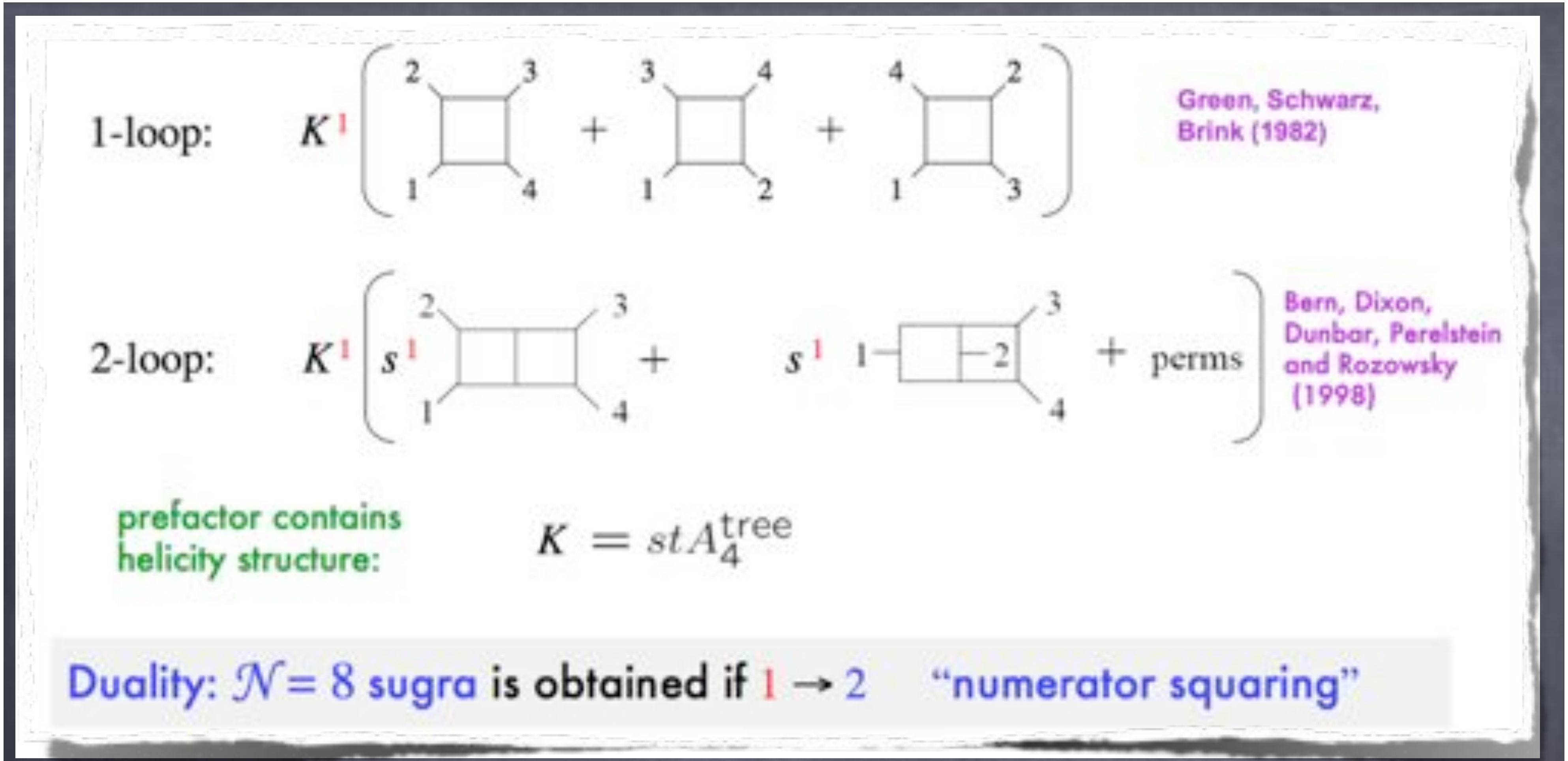
Green, Schwarz, Brink (1982)

2-loop: K^{1} (s^{1} + s^{1} + perms)

Bern, Dixon, Dunbar, Perelstein and Rozowsky (1998)

prefactor contains helicity structure: $K = stA_4^{\text{tree}}$

Duality: $\mathcal{N} = 8$ sugra is obtained if $1 \rightarrow 2$ "numerator squaring"



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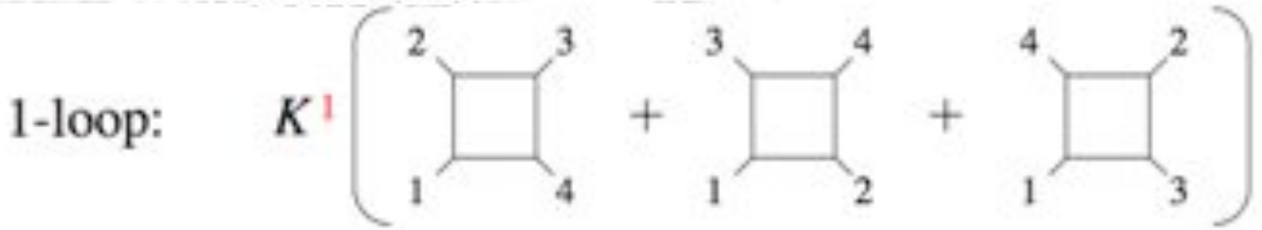
prefactor contains helicity structure: $K = stA_4^{\text{tree}}$

Duality: $\mathcal{N}=8$ sugra is obtained if $1 \to 2$ “numerator squaring”

Aside: Dunbar, Ettle, Perkins have been doing powerful work solving N=4 Sugra all-multiplicity 1-loop MHV using soft and colinear factorizations ‘11,’12 -- wealth of data to try to match to!



Green, Schwarz, Brink (1982)

Note: numerators independent of loop momenta, same true for 5-point 1-loops, so can come out of integrals for double copy

Double copy 1-loop 4&5 point

N<=4 x N=4

=> N>=4 SUGRA

Integrated Expresions

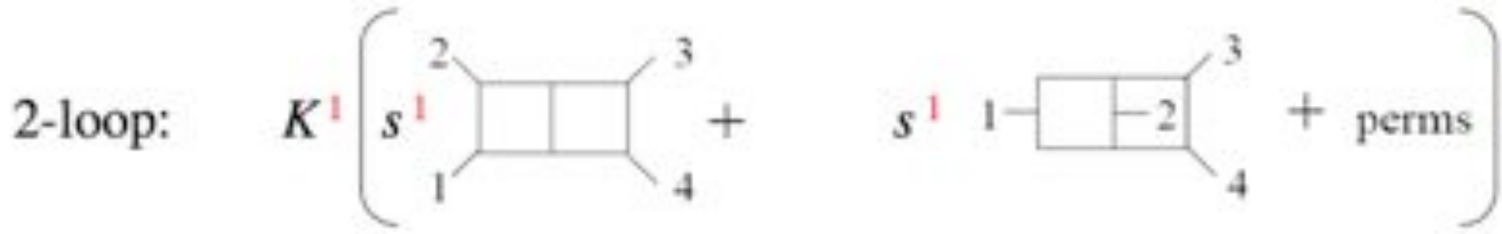
1-loop 4pt Bern, Morgan

1-loop 5pt Bern, Dixon, Kosower

(5pt JJMC, Johansson)

Bern, Boucher-Veronneau, Johansson '11 did the one-loop double-copy reproducing calculations of Dunbar Norridge '96; Dunbar, Ettle, Perkins '10

Aside: Dunbar, Ettle, Perkins have been doing powerful work solving N=4 Sugra all-multiplicity 1-loop MHV using soft and colinear factorizations '11,'12 -- wealth of data to try to match to!



Bern, Dixon, Dunbar, Perelstein and Rozowsky (1998)

Same for 2-loop 4-point

Double copy 2-loop 5 point

N<=4 x N=4

=> N>=4 SUGRA

2-loop 4pt Bern, DeFrietas,Dixon

Boucher-Veronneau, Dixon '11 did the first 2-loop N=4 SUGRA calculation

Very strong checks from IR knowledge: that soft divergences exponentiate

Naculich, Schnitzer; Naculich, Nastase, Schnitzer; White; Brandhuber, Heslop, Nasti, Spence, Travaglini


DEN GODE,
DEN ONDE
OG DEN GRUSOMME

Underlying Algebra?

- **Understanding in 4D in self-dual sector, translating into 4D MHV**

Monteiro, O'Connell

- **Inverting standard color decomposition, analog to tracing over kinematics**

Bern, Dennen

$$\mathcal{A}_m^{\text{tree}} = g^{m-2} \sum_{\sigma} \tau_{(12\ldots m)} A_m^{\text{dual}}(1,2,\ldots,m)$$

Solving the functional relations?

- These loop level calculations work beautifully -- when you figure out the right ansatz!
  - but ... functional equation solving!

"Small problems at the 5-loops level
aren't small problems."

Want to figure out new techniques
of how to solve these guys.

- Tree-level imposition of symmetry provides many of the same challenges
  - A fine playground for some of these techniques

Broedel, JJMC

Extra slides on supergravity finiteness stuff...

Predictions and thoughts on divergences for N=8 SUGRA in D=4

3 loops	Superspace power counting	Deser, Kay (1978) Green, Schwarz, Brink (1982) Howe and Stelle (1989) Marcus and Sagnotti (1985), etc
5 loops	Partial analysis of unitarity cuts; *If* $\mathcal{N} = 6$ harmonic superspace exists; algebraic renormalisation argument	Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998) Howe and Stelle (2003,2009)
6 loops	*If* $\mathcal{N} = 7$ harmonic superspace exists	Howe and Stelle (2003)
7 loops	*If* offshell $\mathcal{N} = 8$ superspace exists; lightcone gauge locality arguments; Algebraic renormalization arguments; $E_{7(7)}$ symmetry.	Grisaru and Siegel (1982); Howe, Stelle and Bossard (2009) Vanhove; Bjornsson, Green (2010) Kiermaier, Elvang, Freedman(2010) Ramond, Kallosh (2010); Biesert, et al (2010)
8 loops	Explicit identification of potential susy invariant counterterm with full non-linear susy and duality.	Kallosh; Howe and Lindström (1981)
9 loops	Assumes Berkovits' superstring non-renormalization theorems carries over to D=4 $\mathcal{N} = 8$ supergravity and extrapolates to 9 loops.	Green, Russo, Vanhove (2006)

Consensus is for valid 7-loop counterterm in D=4, trouble starting at 5-loops

Do we have an example of a valid counterterm that doesn’t vanish for any accepted symmetry reason?

Yes: N = 4 supergravity at three loops in 4 Dimensions

Consensus: valid R^4 divergence exists for N=4 SUGRA in D = 4. Analogous to 7 loop divergence of N = 8 supergravity

Bossard, Howe, Stelle;
Bossard, Howe, Stelle, Vanhove

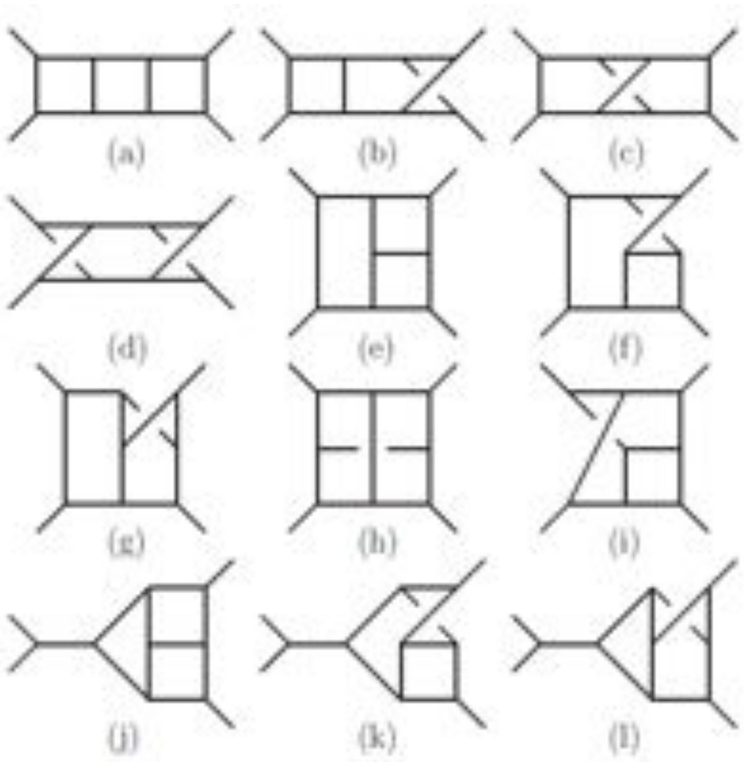
Calculation impossible 2 years ago feasible due to loop-level color-kinematics and double copy

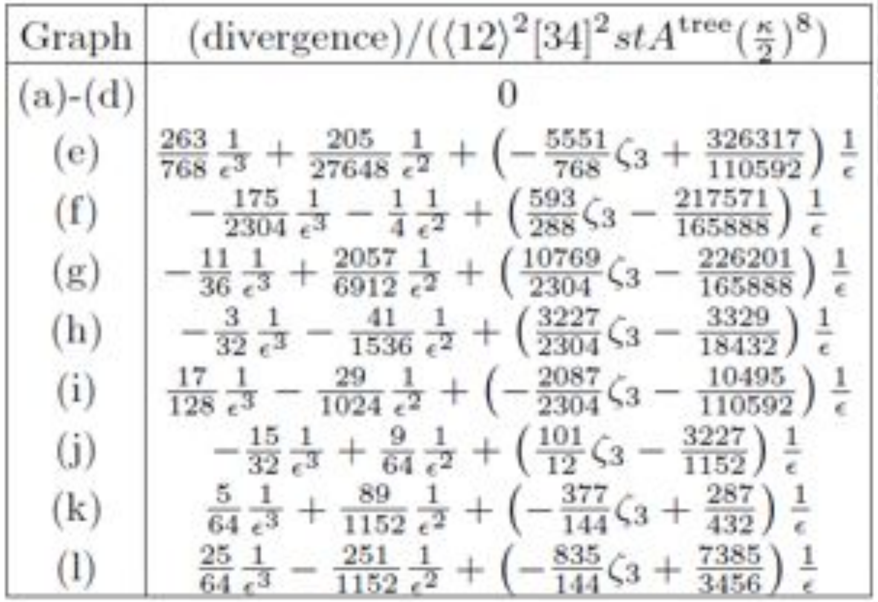
2010 3-loop N=4 SYM CK-rep
x
Feynman Diags for N=0 (QCD)

→ 3-loop
N=4 SUGRA!

The $N = 4$ Supergravity UV Cancellation

ZB, Davies, Dennen, Huang





Graph	(divergence)$/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3}+\frac{205}{27648}\frac{1}{\epsilon^2}+\left(-\frac{5551}{768}\zeta_3+\frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3}-\frac{1}{4}\frac{1}{\epsilon^2}+\left(\frac{593}{288}\zeta_3-\frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3}+\frac{2057}{6912}\frac{1}{\epsilon^2}+\left(\frac{10769}{2304}\zeta_3-\frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3}-\frac{41}{1536}\frac{1}{\epsilon^2}+\left(\frac{3227}{2304}\zeta_3-\frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3}-\frac{29}{1024}\frac{1}{\epsilon^2}+\left(-\frac{2087}{2304}\zeta_3-\frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3}+\frac{9}{64}\frac{1}{\epsilon^2}+\left(\frac{101}{12}\zeta_3-\frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3}+\frac{89}{1152}\frac{1}{\epsilon^2}+\left(-\frac{377}{144}\zeta_3+\frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3}-\frac{251}{1152}\frac{1}{\epsilon^2}+\left(-\frac{835}{144}\zeta_3+\frac{7385}{3456}\right)\frac{1}{\epsilon}$

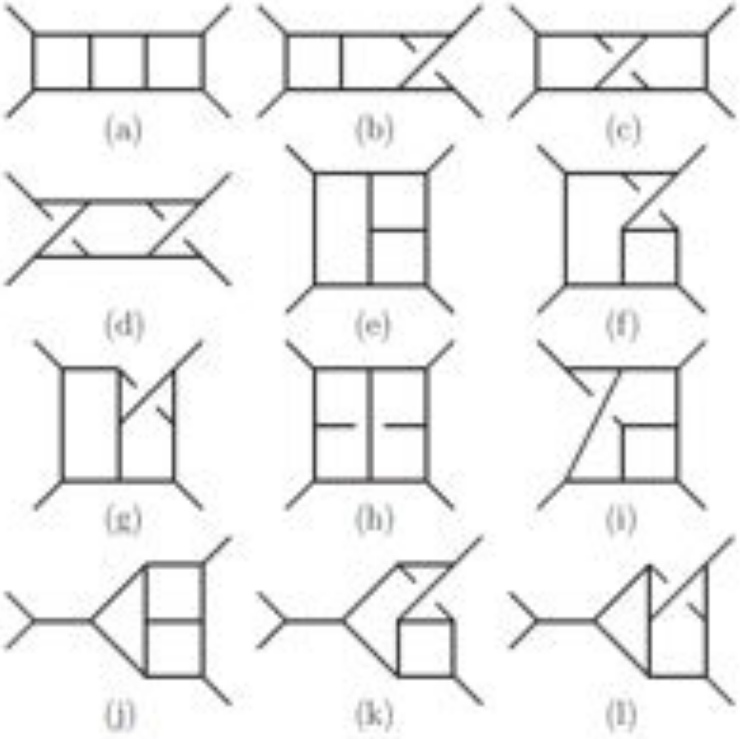
Spinor helicity used to clean up

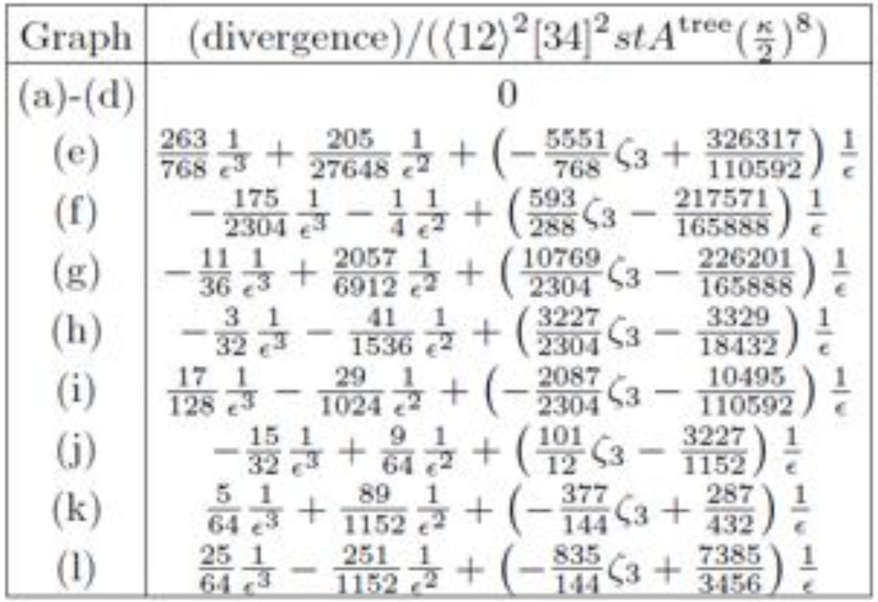
Sum over diagrams is gauge invariant

All divergences cancel completely!

The $\mathcal{N} = 4$ Supergravity UV Cancellation

ZB, Davies, Dennen, Huang





Graph	$(\text{divergence})/(\langle 12\rangle^2[34]^2 st A^{\text{tree}}(\frac{\kappa}{2})^8)$
(a)-(d)	0
(e)	$\frac{263}{768}\frac{1}{\epsilon^3} + \frac{205}{27648}\frac{1}{\epsilon^2} + \left(-\frac{5551}{768}\zeta_3 + \frac{326317}{110592}\right)\frac{1}{\epsilon}$
(f)	$-\frac{175}{2304}\frac{1}{\epsilon^3} - \frac{1}{4}\frac{1}{\epsilon^2} + \left(\frac{593}{288}\zeta_3 - \frac{217571}{165888}\right)\frac{1}{\epsilon}$
(g)	$-\frac{11}{36}\frac{1}{\epsilon^3} + \frac{2057}{6912}\frac{1}{\epsilon^2} + \left(\frac{10769}{2304}\zeta_3 - \frac{226201}{165888}\right)\frac{1}{\epsilon}$
(h)	$-\frac{3}{32}\frac{1}{\epsilon^3} - \frac{41}{1536}\frac{1}{\epsilon^2} + \left(\frac{3227}{2304}\zeta_3 - \frac{3329}{18432}\right)\frac{1}{\epsilon}$
(i)	$\frac{17}{128}\frac{1}{\epsilon^3} - \frac{29}{1024}\frac{1}{\epsilon^2} + \left(-\frac{2087}{2304}\zeta_3 - \frac{10495}{110592}\right)\frac{1}{\epsilon}$
(j)	$-\frac{15}{32}\frac{1}{\epsilon^3} + \frac{9}{64}\frac{1}{\epsilon^2} + \left(\frac{101}{12}\zeta_3 - \frac{3227}{1152}\right)\frac{1}{\epsilon}$
(k)	$\frac{5}{64}\frac{1}{\epsilon^3} + \frac{89}{1152}\frac{1}{\epsilon^2} + \left(-\frac{377}{144}\zeta_3 + \frac{287}{432}\right)\frac{1}{\epsilon}$
(l)	$\frac{25}{64}\frac{1}{\epsilon^3} - \frac{251}{1152}\frac{1}{\epsilon^2} + \left(-\frac{835}{144}\zeta_3 + \frac{7385}{3456}\right)\frac{1}{\epsilon}$

Spinor helicity used to clean up

Sum over diagrams is gauge invariant

All divergences cancel completely!

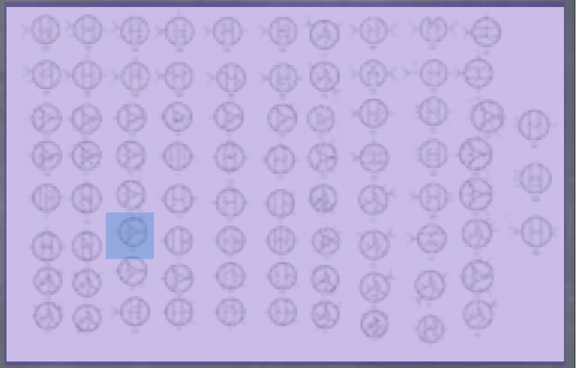
Explanations?

Kallosh '12
Tourkine and Vanhove '12

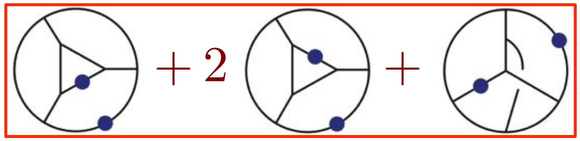
Bern, JJMC, Dixon, Johansson, Roiban (2012)

An interesting development at 4-loops!

In the new manifest representation, we have the power to identify remarkable structure between YM and Gravity



$$\mathcal{M}_4^{(4)}\Big|_{\rm pole} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} stu(s^2+t^2+u^2)^2\, M_4^{\rm tree}\,\Big($$



$$\Big)$$

$$-256 + \frac{2025}{8}$$

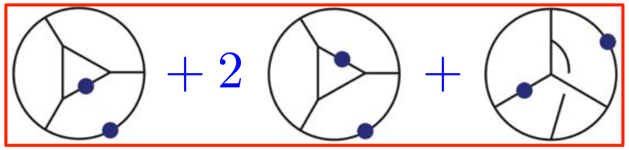
← 12- and 13-propagator integrals

D=11/2

11-propagator integrals; same as in sYM

$$\mathcal{A}_4^{(4)}\Big|^{SU(N_c)}_{\rm pole} = -6\, g^{10}\, \mathcal{K}\, N_c^2 \Big(N_c^2$$

$$+\,12\,\Big($$

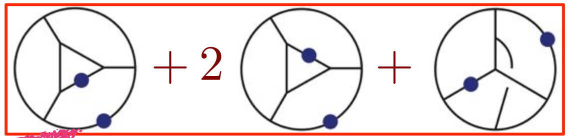


$$\Big)\Big)$$

D=11/2

$$\times \Big(s\,({\rm Tr}_{1324}+{\rm Tr}_{1423}) + t\,({\rm Tr}_{1243}+{\rm Tr}_{1342}) + u\,({\rm Tr}_{1234}+{\rm Tr}_{1432})\Big)$$

Bern, JJMC, Dixon, Johansson, Roiban (2012)



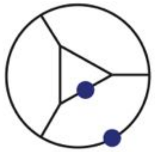
$$\mathcal{M}_4^{(4)}\Big|_{\text{pole}} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} stu(s^2+t^2+u^2)^2\, M_4^{\text{tree}} \left(\quad + 2 \quad + \quad \right)$$

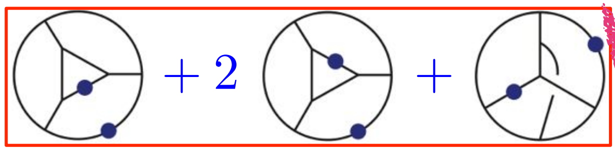
$$\overset{\uparrow}{\color{red}{-256}} \; + \; \frac{2025}{8}$$

12- and 13-propagator integrals

11-propagator integrals; same as in sYM

SAME DIVERGENCE





$$\mathcal{A}_4^{(4)}\Big|^{SU(N_c)}_{\text{pole}} = -6\, g^{10}\, \mathcal{K}\, N_c^2 \left(N_c^2 \quad + 12 \left(\quad + 2 \quad + \quad \right)\right) \times \Big(s\,(\text{Tr}_{1324} + \text{Tr}_{1423}) + t\,(\text{Tr}_{1243} + \text{Tr}_{1342}) + u\,(\text{Tr}_{1234} + \text{Tr}_{1432})\Big)$$

Gravity UV divergence is directly proportional to subleading color single-trace divergence of N = 4 super-Yang-Mills theory.

Same holds for 1-3 loops.

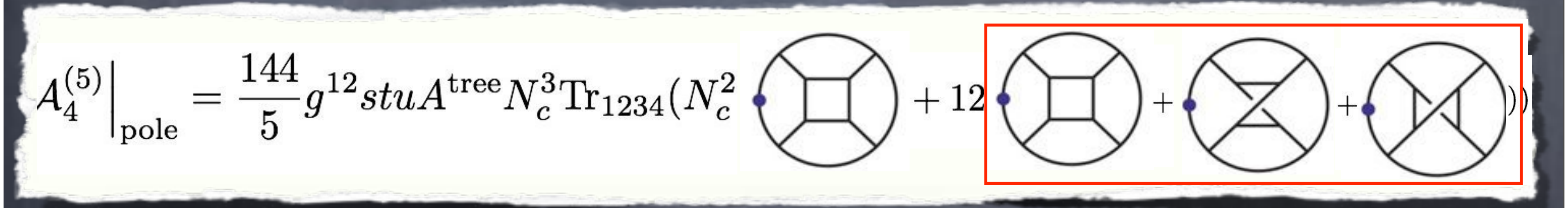
Status of 5-loop SUGRA Calculation

Bern, JJMC, Dixon, Johansson, Roiban

Calculation of N=4 sYM 5-loop Amplitude Complete

arXiv this week?

- Critical step towards getting N=8 5-loop SUGRA Amplitude (working towards finding complete Color-Kinematic satisfying form in progress)
- 416 cubic graphs contributing (in this representation)



$$A_4^{(5)}\Big|_{\text{pole}} = \frac{144}{5} g^{12} stu A^{\text{tree}} N_c^3 \mathrm{Tr}_{1234}(N_c^2 \qquad + 12 \qquad + \qquad + \qquad))$$

No single color-trace terms beyond O(1/Nc^2) suppression (like L<=4)
No double-trace contributions (like L<=4).
Saturates predicted divergence in D=26/5

Clearly if pattern persists for N=8 SUGRA (matching subleading single-trace behavior), N=8 will be UV finite in D=26/5 -- calculation ongoing

Where do we want to end up with these methods?

- Fundamentally rewrite S-matrix so important symmetries and structures can be made manifest - See Arkani-Hamed's talk
- Ok, that may not be immediate, so a **direct** way to write down master(s). (structure constants??)

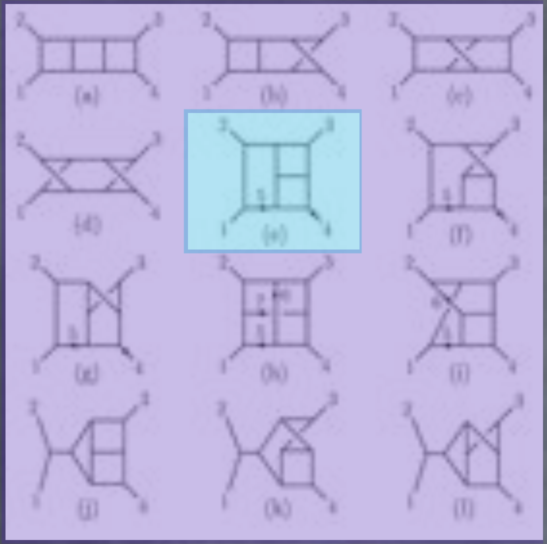
  Bjerrum-Bohr, Damgaard, Monteiro, O'Connell.
- As an intermediate step, we'll be happy with greater control over more **fluidly flowing between representations (c.f. polytopes)**

  Arkani-Hamed, Bourjaily, Cachazo, Hodges, Trnka
- **Generalizations (c.f BLG $n_s = n_t + n_u + n_v$)**

  Bargheer, He, and McLoughlin
- **Existence in higher-genus perturbative string theory?**

  Mafra, Schlotterer, Stieberger
- **What is non-perturbative implication/barrier to understanding gravity as a double-copy?**

Lots to do!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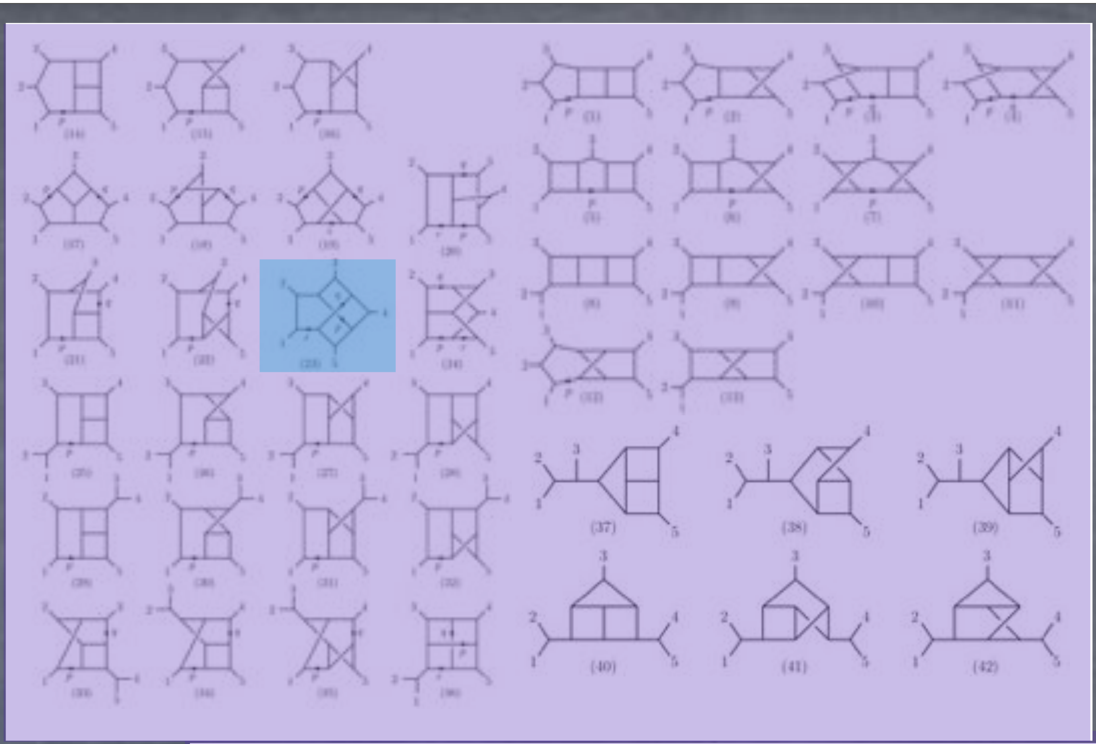
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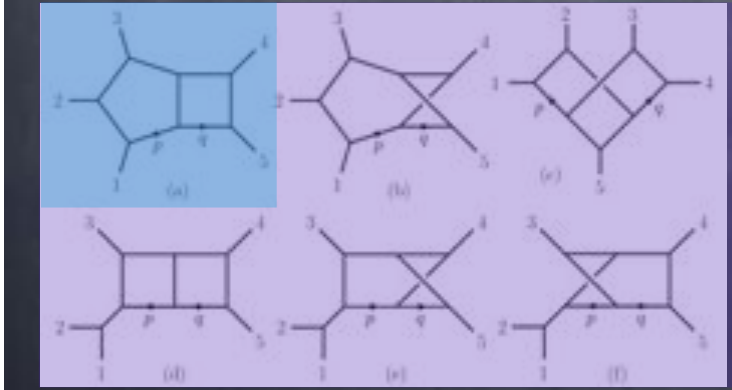
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